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D-branes and SQCD in non-critical superstring theory

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Abstract: Using exact boundary conformal field theory methods we analyze the D-brane physics of a specific four-dimensional non-critical superstring theory which involves the $\mathcal{N} = 2$ SL(2)/U(1) Kazama-Suzuki model at level 1. Via the holographic duality of [1] our results are relevant for D-brane dynamics in the background of NS5-branes and D-brane dynamics near a conifold singularity. We pay special attention to a configuration of D3- and D5-branes that realizes $\mathcal{N} = 1$ supersymmetric QCD and discuss the massless spectrum and classical moduli of this setup in detail. We also comment briefly on the implications of this construction for the recently proposed generalization of the AdS/CFT correspondence by Klebanov and Maldacena within the setting of non-critical superstrings.

Keywords: D-branes, Superstrings and Heterotic Strings, Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

Non-critical superstring theories [2, 3] can be formulated in $d = 2n$ $(n = 0, \ldots, 4)$[1] spacetime dimensions and describe fully consistent solutions of string theory in subcritical dimensions. They have $\mathcal{N} = (2,2)$ worldsheet supersymmetry and appropriate spacetime supersymmetry consisting of (at least) 2^{n+1} spacetime supercharges. On the worldsheet, these theories typically develop a dynamical Liouville mode and they have a target space of the form

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times \mathcal{M}\,, \tag{1.1}$$

[1] $n = 4$ is the critical ten-dimensional fermionic string.

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where $\mathbb{R}_\phi$ is a linear dilaton direction, S^1 is a compact boson and $\mathcal{M}$ is described by a worldsheet theory with $\mathcal{N}=2$ supersymmetry, *e.g.* a Landau-Ginzburg theory or a Gepner product thereof. Due to the linear dilaton, these theories have a strong coupling singularity, which can be resolved in two equivalent ways:

1. We can add to the worldsheet lagrangian a superpotential term of the following form (in superspace language):

$$\delta\mathcal{L} = \mu \int d^2z d^2\theta e^{-\frac{1}{Q}(\phi+iY)} + \text{c.c.} \tag{1.2}$$

   Q denotes the linear dilaton slope, ϕ parametrizes the linear dilaton direction and Y parametrizes the S^1. This interaction couples the $\mathbb{R}_\phi$ and S^1 theories into the well-known $\mathcal{N}=2$ Liouville theory.

2. An alternative way to resolve the strong coupling singularity can be achieved by replacing the $\mathbb{R}_\phi \times S^1$ part of the background (1.1) with the $\mathcal{N}=2$ Kazama-Suzuki supercoset $\mathrm{SL}(2)_k/\,\mathrm{U}(1)$ at level $k=2/Q^2$. This space has a cigar-shaped geometry and provides a geometric cut-off for the strong coupling singularity.

The $\mathcal{N}=2$ Liouville theory and the $\mathcal{N}=2$ Kazama-Suzuki model are known to be equivalent by mirror-symmetry. This non-trivial statement is the supersymmetric version of a similar conjecture in the bosonic case [4] involving the Sine-Liouville theory and the bosonic $\mathrm{SL}(2)/\,\mathrm{U}(1)$ theory. The supersymmetric extension was first conjectured in [5] and later was proven in [6].

Non-critical superstring theories are interesting for a number of reasons. First of all, it has been argued on general grounds [7] that theories with asymptotic linear dilaton directions are holographic. In particular, [1] found that the holographic dual of the d-dimensional theory (1.1) is a corresponding d-dimensional Little String Theory (LST) (for a review see [8, 9]). LST's are non-local, non-gravitational interacting theories that can be defined by taking suitable scaling limits on the worldvolume of NS5-branes or in critical string theory near Calabi-Yau singularities.

LST's appear in various applications. The one that will be the focal point of this paper involves four-dimensional gauge theories that can be realized on D-branes stretched between NS5-branes (for a review of the subject see [10]). A typical brane configuration that realizes four-dimensional $\mathcal{N}=1$ super-Yang-Mills (SYM), say in type-IIA string theory, consists of two NS5-branes and N_c D4-branes oriented as follows (see figure 1):

$$\begin{aligned} NS5: &\ \ (x^0,x^1,x^2,x^3,x^4,x^5) \\ NS5': &\ \ (x^0,x^1,x^2,x^3,x^8,x^9) \\ D4: &\ \ (x^0,x^1,x^2,x^3,x^6)\,. \end{aligned} \tag{1.3}$$

The NS5-branes are tilted with respect to each other breaking supersymmetry by one quarter. The N_c D4-branes stretched between the NS5-branes along the 6-direction break the overall supersymmetry by an additional one-half and realize a gauge theory with four supercharges and gauge group $\mathrm{U}(N_c)$.

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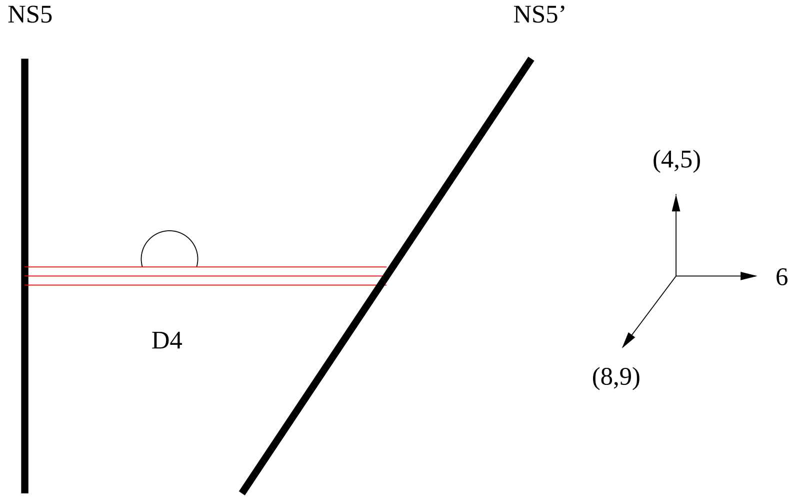

Figure 1: A configuration of two NS5-branes and N_c suspended D4-branes that realizes $\mathcal{N} = 1$ SYM. Flavors can be introduced by adding appropriately oriented D6-branes or semi-infinite D4-branes.

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In order to obtain a truly four-dimensional gauge theory and to decouple the gauge dynamics from other complications of string theory we need to take the double-scaling limit

$$g_s \to 0\,, \qquad L \to 0\,, \qquad g_{\rm YM}^2 = \frac{g_s l_s}{L} = \text{fixed}\,, \tag{1.4}$$

where L is the length of the finite D4-branes in the 6-direction and the limit is taken in such a way that the effective g_{YM} coupling of the gauge theory is kept fixed. This limit is the same as the double scaling limit of LST [5] and, via the holographic duality of [1], the same brane configuration can be realized by taking N_c D3-branes in the non-critical superstring theory

$$\mathbb{R}^{3,1} \times \mathrm{SL}(2)_1/\,\mathrm{U}(1)\,. \tag{1.5}$$

The D3-branes are extended in $\mathbb{R}^{3,1}$ and are localized near the tip of the cigar-shaped target space of $\mathrm{SL}(2)_1/\,\mathrm{U}(1)$. Flavors can also be realized in this setup by adding D4- or D5-branes in (1.5) (see below for explicit constructions). Equivalently, in the original brane configuration of figure 1 flavors can be introduced by adding appropriately oriented D4- or D6-branes (see *e.g.* [10] or figure 4 in section 5 below).

The main purpose of this paper is to analyze the physics of such D-brane configurations in the non-critical superstring (1.5) using exact boundary conformal field theory methods. Similar configurations of D-branes in type-IIB non-critical string theory have been considered recently by Klebanov and Maldacena [11]. The authors of that paper analyzed a configuration of D3-, D5-, and anti-D5-branes[2] in 6-dimensional supergravity and proposed a very interesting generalization of the AdS/CFT correspondence within the con-

[2]The presence of anti-D5-branes in [11] was anticipated on the basis of certain tadpole cancellation conditions. In what follows, we argue that such conditions are automatically satisfied for the D5-branes we formulate and there is no need to introduce anti-D5-branes.

text of non-critical superstrings.[3] The supergravity results pointed towards an $AdS_5 \times S^1$ holographic dual of $\mathcal{N} = 1$ SQCD in the conformal window. The present work adds a different element to this story by analyzing the relevant D3/D5 configuration from the open string theory point of view. This is bound to be useful for analyzing further aspects of the proposed holographic duality. In general, the connection between non-critical strings and four-dimensional gauge theories has long been anticipated [14] and we hope that the present analysis will be relevant for similar investigations of gauge theories in related contexts.

We should mention that a closely related analysis of D-branes in the background of NS5-branes has been performed previously in [15]. This paper analyzed various aspects of the dynamics of D6-branes and semi-infinite D4-branes in the near horizon geometry of NS5-branes with the use of worldsheet techniques and verified several of the expected properties of the gauge theories realized in this setting. Due to important recent progress in the study of the boundary conformal field theory of SL(2)/ U(1) [16 – 22], motivated by the seminal work of [23 – 25], we are now in position to discuss some additional aspects of this story. Most notably, we have a better control on the properties of the D0-branes localized near the tip of the cigar, which lead to the *finite* D4-branes of figure 1. Indeed, we will see how the technology of [16 – 22] yields the full spectrum of open strings stretching on such branes and how we can use it to engineer interesting QCD-like theories. A related analysis of D-branes in the background of NS5-branes using similar techniques has appeared recently in [26].

The layout of this paper is as follows. In section 2, we review the basic characteristics of type-0 and type-II non-critical superstring theory on (1.5), establish our notation and summarize the key features of the closed string spectrum. In section 3, we proceed to analyze the D-brane physics of the theory by using boundary conformal field theory methods, which allow for explicit computations of the cylinder amplitudes and open string spectra. Adapting the existing knowledge on SL(2)/ U(1) D-branes in the current setup we obtain BPS and non-BPS D3-, D4- and D5-branes and discuss their properties. For simplicity, we focus on D-branes with Neumann boundary conditions in all four flat directions of (1.5). In section 4, we discuss general properties of the BPS D3- and D5-branes of the type-IIB theory. We are especially interested in the massless RR couplings of these branes and the presence (or absence) of potential tadpole cancellation conditions. This sets the stage for the main purpose of this paper; the realization of $\mathcal{N} = 1$ SQCD theories on appropriate D-brane setups within non-critical superstring theory. In section 5 we show explicitly, how this can be achieved with a particular D3-D5 setup that realizes the electric description of $\mathcal{N} = 1$ SQCD. Also, we compare the classical symmetries and moduli of the D-brane configuration with those expected from the gauge theory and find agreement as in previous investigations of this subject [10]. In this discussion the Higgsing moduli and the ability (or inability) to formulate the magnetic description of $\mathcal{N} = 1$ SQCD are particularly interesting points, which appear to be alluding to some yet unexplored properties of D-branes on SL(2)/ U(1). We conclude in section 6 with a brief discussion of our results and interesting future prospects related to Seiberg duality and the holographic duality proposed

[3]For relevant discussions and follow-up work in this direction see [12, 13].

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in [11]. Two appendices contain useful information about the properties of the $\mathrm{SL}(2)/\mathrm{U}(1)$ characters and the GSO projected torus partition sum of the four-dimensional non-critical superstring theory.

2. Non-critical superstrings

In this section we review the most prominent features of the closed string sector of the four-dimensional non-critical superstring theory that we want to analyze, establish our notation and present the torus partition function of the type-0 and type-II theories.

2.1 Notation and representation content of the $\mathrm{SL}(2)/\mathrm{U}(1)$ supercoset

The non-trivial part of the worldsheet theory with target space (1.1) is the two-dimensional superconformal theory $\mathrm{SL}(2)_k/\mathrm{U}(1)$ [27]. This theory can be obtained from the supersymmetric $\mathrm{SL}(2,\mathbb{R})$ WZW model at level k by gauging an appropriate $\mathrm{U}(1)$ subgroup (the details of this gauging can be found in various references - see, for example [28]). It has $\mathcal{N}=(2,2)$ worldsheet supersymmetry and central charge

$$\hat{c} = \frac{c}{3} = 1 + \frac{2}{k}\,. \tag{2.1}$$

In general, k can be any positive real number but in this paper we set $k=1$.[4] We want to couple $\mathrm{SL}(2)_k/\mathrm{U}(1)$ to four-dimensional Minkowski space to obtain a Weyl-anomaly free fermionic string. This implies that the total central charge has to be 15, *i.e.*

$$c_{\text{flat}} + c_{\text{coset}} = 15 \;\;\Leftrightarrow\;\; k = 1\,. \tag{2.2}$$

As a sigma-model, $\mathrm{SL}(2)/\mathrm{U}(1)$ describes string propagation on a cigar-shaped two-dimensional manifold [30, 31] with metric

$$ds^2 = k(d\rho^2 + \tanh^2\rho\, d\theta^2)\,, \qquad \theta \sim \theta + 2\pi\,, \tag{2.3}$$

vanishing B-field and varying dilaton

$$\Phi(\rho) = -\log\cosh\rho + \Phi_0\,. \tag{2.4}$$

This background receives α' corrections in the bosonic case [31], but is exact in the supersymmetric case [32, 33], which is the case of interest in this paper. The value of the dilaton Φ_0 at the tip of the cigar is a free tunable parameter. T-duality along the angular direction of the cigar acts non-trivially and the resulting geometry, which naively looks like a trumpet, is described by a closely related $\mathcal{N}=(2,2)$ superconformal field theory - the $\mathcal{N}=2$ Liouville theory [6].

The representation theory of $\mathrm{SL}(2)/\mathrm{U}(1)$ is a useful tool for the analysis of the closed string spectrum and the formulation of D-branes on the cigar geometry (2.3), (2.4). Since we use it heavily in later sections, it is a good idea to review here the basic unitary

[4]The cases with $k>1$ and $k<1$ exhibit interesting differences. See [29] for a recent discussion.

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representations of SL(2)/ U(1) and the corresponding characters. This will also set up our notation. The representations are labeled by the scaling dimension h and the $\mathrm{U}(1)_R$-charge Q. The *unitary* highest-weight representations of the $\mathcal{N}=2$ Kazama-Suzuki model fall into the following three classes [34–36]:[5]

(a) *Continuous representations*: these are non-degenerate representations with

$$h_{j,m} = \frac{-j(j-1)+m^2}{k}\,, \qquad Q_m = \frac{2m}{k}\,, \tag{2.5}$$

and

$$j = \frac{1}{2} + is\,, \quad s \in \mathbb{R}_{\geq 0}\,, \qquad m = r+\alpha\,, \quad r \in \mathbb{Z}\,, \quad \alpha \in [0,1)\,. \tag{2.6}$$

The NS-sector characters read:[6]

$$\mathrm{ch}_c(h_{j,m}, Q_m; \tau, z)\begin{bmatrix}0\\0\end{bmatrix} = q^{h_{j,m}-(\hat{c}-1)/8} y^{Q_m} \frac{\theta[^0_0](\tau,z)}{\eta(\tau)^3}\,, \tag{2.7}$$

where as usual we set $q = e^{2\pi i\tau}$ and $y = e^{2\pi i z}$. $\theta[^a_b](\tau,z)$, with $a,b = 0,1$, are the standard θ-functions whose properties we summarize in appendix A.

(b) *Discrete representations*: these are degenerate representations with[7]

$$j \in \mathbb{R}\,, \qquad 0 < j < \frac{k+2}{2}\,, \qquad r \in \mathbb{Z} \tag{2.8}$$

and

$$h_{j,r} = \frac{-j(j-1)+(j+r)^2}{k}\,, \qquad Q_{j+r} = \frac{2(j+r)}{k}\,, \qquad r \geq 0\,, \tag{2.9}$$

$$h_{j,r} = \frac{-j(j-1)+(j+r)^2}{k} - r - \frac{1}{2}\,, \qquad Q_{j,r} = \frac{2(j+r)}{k} - 1\,, \qquad r < 0\,. \tag{2.10}$$

Notice that $r = 0$ corresponds to chiral primary fields and $r = -1$ to antichiral primary fields. The corresponding NS-sector characters (for any $r \in \mathbb{Z}$) read:

$$\mathrm{ch}_d(h_{j,r}, Q_{j,r}; \tau, z)\begin{bmatrix}0\\0\end{bmatrix} = q^{\frac{-(j-1/2)^2+(j+r)^2}{k}} y^{\frac{2(j+r)}{k}} \frac{1}{1+(-)^b y q^{\frac{1}{2}+r}} \frac{\theta[^0_0](\tau,z)}{\eta(\tau)^3}\,. \tag{2.11}$$

(c) *Identity representations*: These representations are also degenerate and they have quantum numbers $j = 0$, $r \in \mathbb{Z}$ with

$$h_r = \frac{r^2}{k} - r - \frac{1}{2}\,, \qquad Q_r = \frac{2r}{k} - 1\,, \qquad r < 0\,, \tag{2.12}$$

$$h_0 = 0\,, \qquad Q_0 = 0\,, \qquad r = 0\,, \tag{2.13}$$

$$h_r = \frac{r^2}{k} + r - \frac{1}{2}\,, \qquad Q_r = \frac{2r}{k} + 1\,, \qquad r > 0\,. \tag{2.14}$$

---

[5]The representation theory of the $\mathcal{N}=2$ superconformal algebra is an interesting subject on its own [34, 35, 37–39]. In certain cases, $\mathcal{N}=2$ representations exhibit more involved embedding diagrams associated with the appearance of "sub-singular" vectors and the computation of the corresponding characters becomes highly non-trivial. It is commonly believed however that the unitary representations presented here do not suffer from these subtleties. We would like to thank T. Eguchi, M. Gaberdiel, E. Kiritsis, H. Klemm and Y. Sugawara for helpful correspondence on these issues.

[6]The $\widetilde{NS}$-, R- and $\widetilde{R}$-sector characters will be presented below.

[7]This unitarity bound is restricted further in physical theories to $\frac{1}{2} < j < \frac{k+1}{2}$ [5, 40, 41].

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The corresponding NS-sector characters (for any $r \in \mathbb{Z}$) read:

$$\mathrm{ch}_I(h_r, Q_r; \tau, z)\begin{bmatrix}0\\0\end{bmatrix} = q^{-\frac{1}{4k}+\frac{r^2}{k}-r-\frac{1}{2}} y^{\frac{2r}{k}-1} \times$$
$$\times \frac{1-q}{(1+(-)^b y^{-1} q^{-\frac{1}{2}-r})(1+(-)^b y^{-1} q^{\frac{1}{2}-r})} \frac{\theta[^0_0](\tau,z)}{\eta(\tau)^3}\,. \quad (2.15)$$

R-sector characters can be obtained by applying the 1/2-spectral flow operation. To set the notation straight we define the characters

$$\begin{aligned}
\mathrm{ch}_*(*;\tau,z)\begin{bmatrix}0\\0\end{bmatrix} &= \mathrm{Tr}_{\mathrm{NS}}[q^{L_0-\frac{\hat{c}}{8}} y^{J_0}] \\
\mathrm{ch}_*(*;\tau,z)\begin{bmatrix}0\\1\end{bmatrix} &= \mathrm{Tr}_{\widetilde{\mathrm{NS}}}[(-)^F q^{L_0-\frac{\hat{c}}{8}} y^{J_0}] \\
\mathrm{ch}_*(*;\tau,z)\begin{bmatrix}1\\0\end{bmatrix} &= \mathrm{Tr}_{\mathrm{R}}[q^{L_0-\frac{\hat{c}}{8}} y^{J_0}] \\
\mathrm{ch}_*(*;\tau,z)\begin{bmatrix}1\\1\end{bmatrix} &= \mathrm{Tr}_{\widetilde{\mathrm{R}}}[(-)^F q^{L_0-\frac{\hat{c}}{8}} y^{J_0}]\,.
\end{aligned} \quad (2.16)$$

$*$ is an abbreviation for the specific representation and F denotes the total *fermion* number. As a simple illustration, for the continuous representations we obtain the characters

$$\begin{aligned}
\mathrm{ch}_c(h_{j,m}, Q_m;\tau,z)\begin{bmatrix}0\\0\end{bmatrix} &= q^{h_{j,m}-(\hat{c}-1)/8} y^{Q_m} \frac{\theta[^0_0](\tau,z)}{\eta(\tau)^3}\,, \\
\mathrm{ch}_c(h_{j,m}, Q_m;\tau,z)\begin{bmatrix}0\\1\end{bmatrix} &= q^{h_{j,m}-(\hat{c}-1)/8} y^{Q_m} \frac{\theta[^0_1](\tau,z)}{\eta(\tau)^3}\,, \\
\mathrm{ch}_c(h_{j,m+1/2}, Q_{m+1/2};\tau,z)\begin{bmatrix}1\\0\end{bmatrix} &= q^{h_{j,m+\frac{1}{2}}-(\hat{c}-1)/8} y^{Q_{m+\frac{1}{2}}} \frac{\theta[^1_0](\tau,z)}{\eta(\tau)^3}\,, \\
\mathrm{ch}_c(h_{j,m+1/2}, Q_{m+1/2};\tau,z)\begin{bmatrix}1\\1\end{bmatrix} &= q^{h_{j,m+\frac{1}{2}}-(\hat{c}-1)/8} y^{Q_{m+\frac{1}{2}}} \frac{\theta[^1_1](\tau,z)}{\eta(\tau)^3}\,.
\end{aligned} \quad (2.17)$$

The standard $\mathcal{N}=2$ characters presented above generate a continuous spectrum of $\mathrm{U}(1)_R$ charges under the modular transformation $\mathcal{S}: \tau \to -\frac{1}{\tau}$. This feature spoils the requirement of charge integrality imposed by the type-II GSO projection. Hence, it is desirable to construct a different set of "extended" characters that possess integral $\mathrm{U}(1)_R$ charges and at the same time form a closed set under modular transformations. Such characters have been defined in [17] for the cases with rational central charge by taking appropriate sums over integer spectral flows of the standard characters. Adapting the definition of [17] to the present situation of $k=1$ gives the extended characters

$$\chi_c\left(s, m+\frac{a}{2};\tau,z\right)\begin{bmatrix}a\\b\end{bmatrix} = \sum_{n\in\mathbb{Z}} \mathrm{ch}_c(h_{\frac{1}{2}+is, m+\frac{a}{2}+n}, Q_{m+\frac{a}{2}+n};\tau,z)\begin{bmatrix}a\\b\end{bmatrix}, \qquad m = 0\,, \frac{1}{2}\,, \quad (2.18)$$

$$\chi_d\left(j, \frac{a}{2};\tau,z\right)\begin{bmatrix}a\\b\end{bmatrix} = \sum_{n\in\mathbb{Z}} \mathrm{ch}_d(h_{j,\frac{a}{2}+n}, Q_{j,\frac{a}{2}+n};\tau,z)\begin{bmatrix}a\\b\end{bmatrix}, \qquad j = \frac{\ell}{2}\,, \quad \ell = 1,2\,, \quad (2.19)$$

$$\chi_I(\tau,z)\begin{bmatrix}a\\b\end{bmatrix} = \sum_{n\in\mathbb{Z}} \mathrm{ch}_I(h_{\frac{a}{2}+n}, Q_{\frac{a}{2}+n};\tau,z)\begin{bmatrix}a\\b\end{bmatrix}. \quad (2.20)$$

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The $\mathcal{S}$-modular transformation properties of these characters are summarized in appendix A along with a useful set of character identities. The torus partition function receives contributions from the continuous and discrete representations only (see below). The identity characters appear in the open string spectrum of a special class of cigar D-branes.

2.2 Type-0 and type-II non-critical superstring theory on $\mathbb{R}^{3,1} \times \mathrm{SL}(2)/\,\mathrm{U}(1)$

Type-0 and type-II non-critical superstring theory on $\mathbb{R}^{3,1} \times \mathrm{SL}(2)_1/\,\mathrm{U}(1)$ has been examined previously in [42 – 44]. Valuable information about the spectrum of these theories can be obtained by analyzing the torus partition function. This is also useful for implementing appropriate constraints on the boundary states of the theory later on. In general, the one-loop partition sum contains a volume-diverging contribution from continuous representations and a finite contribution from discrete representations. Both contributions can be obtained using recent results on the torus partition function of the bosonic and supersymmetric $\mathrm{SL}(2)/\,\mathrm{U}(1)$ coset in [41, 45, 46, 21, 47]. Here, we present the resulting expressions for $k = 1$ and summarize the basic features of the type-II spectrum. Earlier results on the continuous part of the type-0 and type-II partition function of the non-critical superstring $\mathbb{R}^{3,1} \times \mathrm{SL}(2)_1/\,\mathrm{U}(1)$ have appeared in [42 – 44]. Further details about the type-II GSO projection appear in appendix B.

The one-loop partition sum of the type-0 theories can be obtained by imposing a diagonal GSO projection of the form

$$\begin{aligned} \text{0A}: \quad & (-)^{J_{\text{GSO}}} = (-)^{\bar{J}_{\text{GSO}}}\,, && \text{in the NS-sector}\,, \\ & (-)^{J_{\text{GSO}}} = (-)^{\bar{J}_{\text{GSO}}+1}\,, && \text{in the R-sector}\,, \\ \text{0B}: \quad & (-)^{J_{\text{GSO}}} = (-)^{\bar{J}_{\text{GSO}}}\,, \end{aligned} \tag{2.21}$$

and the same fermion boundary conditions on the left- and right-moving fermions. The precise definitions of J_{GSO} and $\bar{J}_{\text{GSO}}$ appear in appendix B and include a sum on the fermion number of the flat $\mathbb{R}^{3,1}$ conformal field theory and the $\mathrm{U}(1)_R$ charge of the supercoset. The resulting one-loop partition sum takes the form

$$\begin{aligned} Z_{\text{0A/B}}(\tau,\bar{\tau}) = \frac{1}{2} \sum_{a,b=0,1} \sum_{w\in\mathbb{Z}_2} (-)^{\eta ab} \times \\ \times \bigg\{ \int_0^\infty ds\sqrt{2}\rho(s,w,a;\epsilon)\chi_c\Big(s,\frac{w+a}{2};\tau,0\Big)\begin{bmatrix}a\\b\end{bmatrix}\chi_c\Big(s,\frac{w+a}{2};\bar{\tau},0\Big)\begin{bmatrix}a\\b\end{bmatrix} + \\ + \frac{1}{2}\chi_d\Big(\frac{w}{2},\frac{a}{2};\tau,0\Big)\begin{bmatrix}a\\b\end{bmatrix}\chi_d\Big(\frac{w}{2},\frac{a}{2};\bar{\tau},0\Big)\begin{bmatrix}a\\b\end{bmatrix}\bigg\} \frac{|\theta[^a_b]|^2}{(8\pi^2\tau_2)^2|\eta|^6}\,, \end{aligned} \tag{2.22}$$

with spectral density

$$\rho(s,w,a;\epsilon) = \frac{1}{\pi}\log\epsilon + \frac{1}{4\pi i}\frac{d}{ds}\log\left\{\frac{\Gamma(\frac{1}{2}-is+\frac{a+w}{2})\Gamma(\frac{1}{2}-is-\frac{a+w}{2})}{\Gamma(\frac{1}{2}+is+\frac{a+w}{2})\Gamma(\frac{1}{2}+is-\frac{a+w}{2})}\right\}. \tag{2.23}$$

In this expression ϵ denotes the IR cutoff that regularizes the infinite volume divergence of the cigar CFT. $\eta = 0/1$ corresponds to the type-0B/0A theory.

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One can easily check that the volume diverging piece of this partition sum is identical to the one appearing in eq. (B.10) of [44]. The extra discrete piece is a by-product of the analysis appearing in refs. [41, 45−47]. In our case ($k = 1$), there are no discrete characters with half-integer j inside the interval $\mathcal{J} := (\frac{1}{2}, \frac{k+1}{2} = 1)$ and the only discrete characters appearing in (2.22) are those lying on the boundaries of $\mathcal{J}$. This extra contribution arises by defining the integral over the continuous parameter s with a principal value prescription that singles out a pole at $s = 0$ (for a nice exposition of the relevant details see [46]).

To obtain the one-loop partition sum of the type-II theory one should perform a two-step procedure:

(i) Impose the condition of integral $\mathrm{U}(1)_R$ charges. This condition is necessary for a well-defined chiral GSO projection in step (ii) below. In the torus partition sum (2.22) this integrality condition is automatic. Indeed, the characters appearing in the type-0A/B partition sum have integral coset $\mathrm{U}(1)_R$ charges in the NS-sector

$$Q = 2\frac{w}{2} = w \in \mathbb{Z}_2 \tag{2.24}$$

and the total fermion number is always an integer (see appendix B for further details).

(ii) Perform the chiral GSO projection. On the level of vertex operators this projection requires mutual locality with respect to the spacetime supercharges of the theory and, similar to the ten-dimensional critical case, it leads ultimately to a type-IIA or type-IIB theory. In the non-critical case this prescription has a peculiar feature (this point was emphasized in [44]). It leads to a non-trivial coupling of the spin of the particles with their momentum around the angular direction of the cigar and gives a spectrum that does not have a natural spacetime interpretation as particles propagating in six-dimensional curved spacetime. Instead, the theory has a natural holographic interpretation as a non-gravitational theory living in four dimensions.

Implementing the above procedure yields the following one-loop partition sum

$$\begin{aligned} Z_{\mathrm{II}}(\tau,\bar\tau) &= \frac{1}{4}\sum_{a,\bar a,b,\bar b=0,1}\ \sum_{w\in\mathbb{Z}_2}(-)^{\eta ab+a+\bar a+(w+1)(b+\bar b)}\times \\ &\times\Bigg\{\int_0^\infty ds\sqrt{2}\rho(s,w;a,\bar a;\epsilon)\chi_c\left(s,\frac{w+a}{2};\tau,0\right)\begin{bmatrix}a\\b\end{bmatrix}\chi_c\left(s,\frac{w+\bar a}{2};\bar\tau,0\right)\begin{bmatrix}\bar a\\ \bar b\end{bmatrix}+ \\ &+\frac{1}{2}\chi_d\left(\frac{w}{2},\frac{a}{2};\tau,0\right)\begin{bmatrix}a\\b\end{bmatrix}\chi_d\left(\frac{w}{2},\frac{\bar a}{2};\bar\tau,0\right)\begin{bmatrix}\bar a\\ \bar b\end{bmatrix}\Bigg\}\frac{1}{(8\pi^2\tau_2)^2\eta^2\bar\eta^2}\frac{\theta[^a_b]}{\eta}\frac{\theta[^{\bar a}_{\bar b}]}{\bar\eta}, \end{aligned} \tag{2.25}$$

where

$$\rho(s,w;a,\bar a;\epsilon) = \frac{1}{\pi}\log\epsilon + \frac{1}{4\pi i}\frac{d}{ds}\log\left\{\frac{\Gamma(\frac{1}{2}-is+\frac{a+w}{2})\Gamma(\frac{1}{2}-is-\frac{\bar a+w}{2})}{\Gamma(\frac{1}{2}+is+\frac{a+w}{2})\Gamma(\frac{1}{2}+is-\frac{\bar a+w}{2})}\right\}. \tag{2.26}$$

Again, one can check that the volume-diverging piece of this partition sum is identical to the one appearing in [42] or [44] (see eq. (B.13) of the latter paper). By supersymmetry, we expect (2.25) to be zero because of the exact cancellation between bosons and fermions.

Theory	Sector	Fields
IIA and IIB	$NS+NS+$	$G_{\mu\nu},\ B_{\mu\nu},\ \phi$
	$NS-NS-$	$T,\ T'$
IIA	$R+R-$	A_1
	$R-R+$	A'_1
IIB	$R+R+$	$C_0,\ C_2^+$
	$R-R-$	$C'_0,\ C_2^-$

Table 1: The bosonic spectrum of type-IIA and type-IIB non-critical superstring theory in (1.5). The plus or minus superscripts for the RR potentials denote the self-dual or anti-selfdual part respectively. The subscript denotes the rank of the corresponding field. The fermionic part of the spectrum (NS-R sectors) follows trivially by supersymmetry.

Indeed, we can check this explicitly for the continuous contributions by writing everything in terms of the character combinations

$$\begin{aligned}\Lambda_1(s;\tau) \ =\ & \left(\chi_c(s,0;\tau,0)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau,0)}{\eta(\tau)^3} - \chi_c(s,0;\tau,0)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](\tau,0)}{\eta(\tau)^3}\right) - \\ & -\left(\chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau,0)}{\eta(\tau)^3} - \chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau,0)}{\eta(\tau)^3}\right), \qquad (2.27)\\ \Lambda_{-1}(s;\tau) \ =\ & \left(\chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau,0)}{\eta(\tau)^3} + \chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](\tau,0)}{\eta(\tau)^3}\right) - \\ & -\left(\chi_c(s,0;\tau,0)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau,0)}{\eta(\tau)^3} + \chi_c(s,0;\tau,0)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau,0)}{\eta(\tau)^3}\right). \qquad (2.28)\end{aligned}$$

These combinations are known to be zero identically [48, 49]. To check the vanishing of the discrete contributions one has to use in addition the results of appendix A.

A few comments on the closed string spectrum. Closing this section we would like to make a few final remarks on the closed string spectrum following from the torus partition function (2.25). A summarizing list of (the bosonic part of) this spectrum from the six-dimensional point of view appears in table 1.

The majority of fields appearing in this table are massive. For instance, all the fields appearing in the NS+NS+ sector are massive including the graviton. Massless fields arise from (continuous or discrete) representations with $j = \frac{1}{2}$ in the NS−NS− and R+R+ sectors (for simplicity we discuss only the bosonic sector here - the fermionic sector can be determined easily by supersymmetry). More precisely, from the NS−NS− sector we obtain two massless complex tachyons T, T'. One of them has winding number $|w| = 1$ and momentum zero and the other has winding number zero and momentum $|n| = 1$. Physical massless states in the RR sector are (from the six-dimensional point of view) in the $\mathbf{2}\times\mathbf{2} = [0] + [2]_+$ representation of the little group SO(4) for the type-IIB theory and in the $\mathbf{2}\times\mathbf{2}' = [1]$ for the type-IIA theory. In the type-IIB case they correspond to a scalar C_0 and a self-dual 2-form C_2^+. In the type-IIA case they correspond to a vector A_1. In both cases, these fields reduce to two scalars and one vector in four dimensions, as expected from the unique non-chiral structure of four-dimensional $\mathcal{N} = 2$ supersymmetry.

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3. Boundary conformal field theory on $\mathbb{R}^{3,1} \times \mathrm{SL}(2)/\mathrm{U}(1)$

In superstring theory it is standard to impose boundary conditions preserving at least $\mathcal{N}=1$ superconformal invariance on the boundary of the worldsheet. In the closed string channel this implies boundary conditions of the form

$$\begin{aligned}(L_n - \bar{L}_{-n})|B\rangle &= 0\,, \\ (G_r - i\eta \bar{G}_{-r})|B\rangle &= 0\,, \end{aligned} \tag{3.1}$$

where $\eta = \pm 1$ denotes the spin structure of the fermionic generators.

In the flat $\mathbb{R}^{3,1}$ part of our theory these conditions can be satisfied in the standard way familiar from ten-dimensional critical superstring theory [50 − 52]. In later parts of this paper we want to consider D-brane configurations that realize a $(3+1)$-dimensional gauge theory. Hence, we have to impose Neumann boundary conditions in all four flat directions of $\mathbb{R}^{3,1} \times \mathrm{SL}(2)/\mathrm{U}(1)$ and the corresponding Ishibashi states will be characterized by a vanishing momentum and the spin structure of the fermions. These states will be denoted simply as

$$|p_\mu = 0; \begin{bmatrix} a \\ b \end{bmatrix}\Big\rangle\Big\rangle \equiv \Big|\begin{bmatrix} a \\ b \end{bmatrix}\Big\rangle\Big\rangle_{\rm flat} \tag{3.2}$$

and they have a standard construction as coherent states in the free supersymmetric $\mathbb{R}^{3,1}$ conformal field theory. In the covariant formalism, which is the formalism we are implicitly adopting, one should include also the contribution of ghosts. The explicit form of the ghost boundary states can be found in [50]. In (3.2) the label $a = 0, 1$ parametrizes a boundary state in the NSNS and RR sectors respectively, while the second label $b = 0, 1$ parametrizes the choice of spin structure η. The corresponding cylinder amplitudes take the form

$${}_{\rm flat}\Big\langle\Big\langle \begin{bmatrix} a' \\ b' \end{bmatrix}\Big| e^{-\pi T H^c_{\rm flat}} \Big|\begin{bmatrix} a \\ b \end{bmatrix}\Big\rangle\Big\rangle_{\rm flat} = (-)^a \delta_{a,a'} \frac{\theta[{}^{\;a}_{b-b'}](iT,0)}{\eta^3(iT)}\,. \tag{3.3}$$

In $\mathrm{SL}(2)/\mathrm{U}(1)$ we choose to impose a more symmetric set of boundary conditions preserving $\mathcal{N}=2$ superconformal invariance on the boundary of the worldsheet. These are the well-known boundary conditions [53]:

$$\mathrm{A-type:}\;\; (J_n - \bar{J}_{-n})|B\rangle = 0\,, \qquad (G^\pm_r - i\eta \bar{G}^\mp_{-r})|B\rangle = 0\,, \tag{3.4}$$
$$\mathrm{B-type:}\;\; (J_n + \bar{J}_{-n})|B\rangle = 0\,, \qquad (G^\pm_r - i\eta \bar{G}^\pm_{-r})|B\rangle = 0\,. \tag{3.5}$$

The A-type boundary conditions are Neumann in the angular direction of the cigar and the B-type are Dirichlet.[8] Corresponding Ishibashi states can be constructed based on continuous or discrete representations. These will be denoted as $|X; s, m, \bar{m}; [{}^a_b]\rangle\rangle_{\rm cos}$ for the continuous representations and $|X; j; [{}^a_b]\rangle\rangle_{\rm cos}$ for the discrete. $X = A$, B is an extra label specifying the type of boundary condition and the parameters $s, m, \bar{m}, j$ take the

[8] In more standard conventions (*c.f.* [17, 19]) A-type boundary conditions are always Dirichlet and B-type are Neumann. In this paper we use the opposite convention associated with the right-moving $\mathcal{N}=2$ current $\bar{J}_{\mathcal{N}=2} = \bar{\psi}^+\bar{\psi}^- + \frac{2}{k}\bar{\mathcal{J}}^3$.

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appropriate values dictated by the representations appearing in the torus partition sum and the specific boundary conditions. The corresponding cylinder amplitudes are

$$
\begin{aligned}
{}_{\mathrm{cos}}\left\langle\left\langle X; s, m, \bar{m}; \begin{bmatrix} a \\ b \end{bmatrix}\right| e^{-\pi T H^c_{\mathrm{coset}}} \left| X; s', m', \bar{m}'; \begin{bmatrix} a' \\ b' \end{bmatrix}\right\rangle\right\rangle_{\mathrm{cos}} &= \\
&= \delta_{a,a'}\delta(s-s')\delta_{m,m'}\chi_c(s,m;iT,0)\begin{bmatrix} a \\ b'-b \end{bmatrix}, \\
{}_{\mathrm{cos}}\left\langle\left\langle X; j; \begin{bmatrix} a \\ b \end{bmatrix}\right| e^{-\pi T H^c_{\mathrm{coset}}} \left| X; j'; \begin{bmatrix} a' \\ b' \end{bmatrix}\right\rangle\right\rangle_{\mathrm{cos}} &= \\
&= \delta_{a,a'}\delta_{j,j'}\chi_d(j,\frac{a}{2};iT,0)\begin{bmatrix} a \\ b'-b \end{bmatrix}.
\end{aligned}
\tag{3.6}
$$

The Ishibashi states of the full theory are tensor products of the $\mathbb{R}^{3,1}$ Ishibashi states $|[{a \atop b}]\rangle\rangle_{\mathrm{flat}}$ with A- or B-type Ishibashi states of the coset. However, the generic tensor product is not an allowed Ishibashi state. Only those states that couple to the closed string modes appearing in the torus partition sum (2.25) are allowed. This implies a set of constraints.

First, we have a constraint on the combination of spin structures. The same spin structure must appear on the flat and coset components, *i.e.* we should restrict to boundary states of the form

$$
\left| X; s, m, \bar{m}; \begin{bmatrix} a \\ b \end{bmatrix}\right\rangle\right\rangle = \left|\begin{bmatrix} a \\ b \end{bmatrix}\right\rangle\right\rangle_{\mathrm{flat}} \otimes \left| X; s, m, \bar{m}; \begin{bmatrix} a \\ b \end{bmatrix}\right\rangle\right\rangle_{\mathrm{cos}}
\tag{3.7}
$$

and

$$
\left| X; j; \begin{bmatrix} a \\ b \end{bmatrix}\right\rangle\right\rangle = \left|\begin{bmatrix} a \\ b \end{bmatrix}\right\rangle\right\rangle_{\mathrm{flat}} \otimes \left| X; j; \begin{bmatrix} a \\ b \end{bmatrix}\right\rangle\right\rangle_{\mathrm{cos}}.
\tag{3.8}
$$

This can be rephrased as the requirement to have a well-defined periodicity for the total $\mathcal{N}=1$ supercurrent $G_{\mathrm{total}} = G_{\mathrm{flat}} + G^{+}_{\mathrm{coset}} + G^{-}_{\mathrm{coset}}$.

A second set of constraints comes from GSO invariance. For simplicity, let us consider here only the type-IIB case. By simple inspection of the torus partition sum (2.25), or by explicitly checking how $(-)^{J_{\mathrm{GSO}}}$, $(-)^{\bar{J}_{\mathrm{GSO}}}$ act on the Ishibashi states and requiring $(-)^{J_{\mathrm{GSO}}} = (-)^{\bar{J}_{\mathrm{GSO}}} = 1$, we find a set of GSO-allowed linear superpositions of Ishibashi states. For example, the allowed NSNS continuous Ishibashi states are

$$
\begin{aligned}
\left| A; s, 0, 0; + \right\rangle\right\rangle_{NS} &= \left| A; s, 0, 0; \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\rangle\right\rangle - \left| A; s, 0, 0; \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\rangle\right\rangle, \\
\left| A; s, \frac{1}{2}, \frac{1}{2}; - \right\rangle\right\rangle_{NS} &= \left| A; s, \frac{1}{2}, \frac{1}{2}; \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\rangle\right\rangle + \left| A; s, \frac{1}{2}, \frac{1}{2}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\rangle\right\rangle.
\end{aligned}
\tag{3.9}
$$

Notice the correlation between the quantum numbers $m, \bar{m}$ and the sign of total fermion chirality $(-)^{F_{\mathrm{fermion}}+a-1}$, which appears as an extra index $\pm$ in the Ishibashi state. The corresponding RR sector Ishibashi states take the form

$$
\begin{aligned}
| A; s, 0, 0; + \rangle\rangle_{R} &= \left| A; s, 0, 0; \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\rangle\right\rangle + \left| A; s, 0, 0; \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\rangle\right\rangle, \\
\left| A; s, \frac{1}{2}, \frac{1}{2}; - \right\rangle\right\rangle_{R} &= \left| A; s, \frac{1}{2}, \frac{1}{2}; \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\rangle\right\rangle - \left| A; s, \frac{1}{2}, \frac{1}{2}; \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\rangle\right\rangle.
\end{aligned}
\tag{3.10}
$$

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The flip of sign conventions between the NSNS and RR sectors is due to the superconformal ghost contribution to $(-)^{F_{\text{fermion}}+a-1}$. Similar expressions can be written for the A-type discrete states and for the B-type NSNS Ishibashi states. The B-type RR Ishibashi states have $(-)^{J_{\text{GSO}}} = -(-)^{\bar{J}_{\text{GSO}}} = 1$ and they have to be excluded in type-IIB string theory. This point has important consequences for the BPS spectrum of branes in this theory and we would like to explain it here in some detail.

Working in the covariant formalism we can write the full GSO charge in the Ramond sector as

$$J_{\text{GSO}} = F_{\text{flat}} + J_{\mathcal{N}=2} - \frac{1}{2} \tag{3.11}$$

and this should be an even integer for GSO projected states. The last term $-\frac{1}{2}$ comes from the superghost contribution. F_{flat} denotes the flat space fermion number

$$F_{\text{flat}} = s_0 + s_1\,, \qquad s_0,\ s_1 = \pm\frac{1}{2} \tag{3.12}$$

and $J_{\mathcal{N}=2}$ is the $\mathrm{U}(1)_R$ charge

$$J_{\mathcal{N}=2} = 2m_R + \frac{1}{2}\,. \tag{3.13}$$

The half-integer m_R is the R-sector $\mathcal{J}^3$ charge of $\mathrm{SL}(2)/\,\mathrm{U}(1)$. For B-type boundary conditions the right-moving charges are related to the left ones by the following equations

$$F_{\text{flat}} = \bar{F}_{\text{flat}}\,, \qquad J_{\mathcal{N}=2} = -\bar{J}_{\mathcal{N}=2}\,. \tag{3.14}$$

Hence,

$$\bar{J}_{\text{GSO}} = \bar{F}_{\text{flat}} + \bar{J}_{\mathcal{N}=2} - \frac{1}{2} = s_0 + s_1 - 2m_R - 1 \tag{3.15}$$

and

$$(-)^{\bar{J}_{\text{GSO}}} = (-)^{s_0+s_1-2m_R-1} = -(-)^{J_{\text{GSO}}} \tag{3.16}$$

as claimed above.

Implementing the full set of the above constraints we find the allowed Ishibashi states

- *A-type, continuous:*

$$\begin{aligned} &|A; s, 0, 0; +\rangle\rangle_{NS}\,, \qquad \left|A; s, \frac{1}{2}, \frac{1}{2}; -\right\rangle\Big\rangle_{NS}\,, \\ &|A; s, 0, 0; +\rangle\rangle_{R}\,, \qquad \left|A; s, \frac{1}{2}, \frac{1}{2}; -\right\rangle\Big\rangle_{R}\,, \qquad s \in \mathbb{R}_+\,, \end{aligned} \tag{3.17}$$

- *B-type, continuous:*

$$|B; s, 0, 0; +\rangle\rangle_{NS}\,, \qquad \left|B; s, \frac{1}{2}, -\frac{1}{2}; -\right\rangle\Big\rangle_{NS}\,, \qquad s \in \mathbb{R}_+\,. \tag{3.18}$$

Similar discrete A-type Ishibashi states exist, but they will not be mentioned here explicitly, since they play no role in the boundary state analysis of the next subsections.

In what follows we employ these results to formulate and analyze the properties of D-branes in the four-dimensional non-critical superstring theory under consideration.

3.1 A-type boundary states

In this subsection we formulate A-type boundary states as appropriate linear combinations of the Ishibashi states presented above. The coefficients can be determined by using previously obtained results on the boundary states of the coset SL(2)/ U(1). In some cases these coefficients follow directly from a generalized Cardy ansatz, but there are also situations where one has to use slight variants that have been derived by different methods. Here we discuss each case in detail and explain any potential subtleties. At the end, we verify the Cardy consistency conditions by a straightforward computation of the annulus amplitudes.

A generic A-type boundary state labelled by ξ will be written in the NS and R-sectors as

$$|A;\xi\rangle\rangle_{NS} = \int_0^\infty ds \left(\Phi_{NS}(s,+;\xi)|A;s,0;+\rangle\rangle_{NS} + \Phi_{NS}(s,-;\xi)\left|A;s,\frac{1}{2};-\right\rangle\right\rangle_{NS}\right), \quad (3.19)$$

$$|A;\xi\rangle\rangle_{R} = \int_0^\infty ds \left(\Phi_{R}(s,+;\xi)|A;s,0;+\rangle\rangle_{R} + \Phi_{R}(s,-;\xi)\left|A;s,\frac{1}{2};-\right\rangle\right\rangle_{R}\right). \quad (3.20)$$

ξ will be an index or a set of indices characterizing the SL(2)/ U(1) properties of the brane. In principle, ξ can be a label corresponding to continuous, discrete or identity representations, but a more precise analysis reveals the following possibilities.

Class 1. Boundary states in this class are based on the *identity* representation and will be denoted as $|A\rangle_{NS}$ and $|A\rangle_R$. They can be obtained from a direct application of the Cardy ansatz, which implies in our case the following wavefunctions[9]

$$\Phi_{NS}(s,+;I) = \Phi_R(s,-;I) = \frac{1}{2}\sqrt{S^c\left(s,0;\begin{bmatrix}0\\0\end{bmatrix}\middle|I;\begin{bmatrix}0\\0\end{bmatrix}\right)} = \sinh(\pi s)\,, \quad (3.21)$$

$$\Phi_{NS}(s,-;I) = \Phi_R(s,+;I) = \frac{1}{2}\sqrt{S^c\left(s,\frac{1}{2};\begin{bmatrix}0\\0\end{bmatrix}\middle|I;\begin{bmatrix}0\\0\end{bmatrix}\right)} = \cosh(\pi s)\,. \quad (3.22)$$

The corresponding reflection-invariant one-point functions on the disc are

$$\begin{aligned}
\langle \mathcal{V}^{NS+NS+}_{\frac{1}{2}+is,m,\bar m}(p^\mu)\rangle &= \langle \mathcal{V}^{R-R-}_{\frac{1}{2}+is,m+\frac{1}{2},\bar m+\frac{1}{2}}(p^\mu)\rangle \\
&= \delta^{(4)}(p^\mu)\delta_{m,\bar m}\frac{1}{2}\frac{\Gamma(\frac{1}{2}+is+m)\Gamma(\frac{1}{2}+is-m)}{\Gamma(1+2is)\Gamma(2is)} \\
&\sim \delta^{(4)}(p^\mu)\delta_{m,\bar m}\sinh(\pi s)\,, \qquad m\in\mathbb{Z}\,, \qquad (3.23)\\
\langle \mathcal{V}^{NS-NS-}_{\frac{1}{2}+is,m,\bar m}(p^\mu)\rangle &= \langle \mathcal{V}^{R+R+}_{\frac{1}{2}+is,m+\frac{1}{2},\bar m+\frac{1}{2}}(p^\mu)\rangle \\
&= \delta^{(4)}(p^\mu)\delta_{m,\bar m}\frac{1}{2}\frac{\Gamma(\frac{1}{2}+is+m)\Gamma(\frac{1}{2}+is-m)}{\Gamma(1+2is)\Gamma(2is)} \\
&\sim \delta^{(4)}(p^\mu)\delta_{m,\bar m}\cosh(\pi s)\,, \qquad m\in\mathbb{Z}+\frac{1}{2}\,. \qquad (3.24)
\end{aligned}$$

[9] Here and below we do not include a standard phase factor ν_k^{is}, with $\nu_k = \frac{\Gamma(1-\frac{1}{k})}{\Gamma(1+\frac{1}{k})}$, because it diverges for $k=1$. This factor does not affect the computation of annulus amplitudes.

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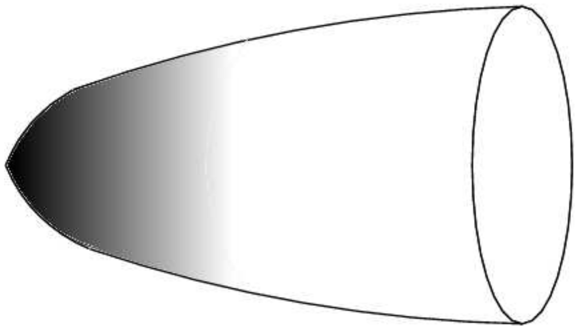

Figure 2: D3-branes have a smooth profile in the radial direction of the cigar supported near the tip.

The similarity symbol $\sim$ denotes equality up to a phase and p^μ is the four-dimensional Minkowski space momentum.

These boundary states correspond to D3-branes and can be thought of as the analogs of the Liouville theory ZZ-branes. Geometrically, they are localized near the tip of the cigar (see figure 2) with a smooth profile along the radial direction. In general, there are two clear signals of the localization of this class of branes near the tip: the vanishing of some of the continuous wavefunctions for zero radial momentum s and the presence of discrete couplings. The first property is apparent in (3.21), but the second is not as a consequence of the very special features of the $k = 1$ case.

Class 2. In this class we consider boundary states based on the *continuous* representations. They will be denoted as $|A; s, m\rangle_{NS}$ and $|A; s, m\rangle_R$, with parameters $s \in \mathbb{R}_{\geq 0}$ and $m = 0, \frac{1}{2}$. On $\mathrm{SL}(2)_k/\mathrm{U}(1)$ (for even levels k) these branes were first formulated in [21]. There it was argued that they correspond to D2-branes partially or totally covering the cigar, where s is a modulus parametrizing the closest distance between the brane and the tip (for the semiclassical analysis of these branes see [54]).

The precise form of their wavefunctions (for generic integer level k) can be determined in the following way. Starting from the T-dual trumpet geometry, which strictly speaking is described by the $\mathcal{N} = 2$ Liouville theory, we can formulate B-type D1-branes which extend in the radial direction. The expressions and consistency of the wavefunctions of the corresponding boundary states has been determined by direct computation with modular and conformal bootstrap methods in [22].[10] After a T-duality transformation the resulting expressions for the A-type class 2 cigar boundary states at $k = 1$ are:

$$\begin{aligned}\Phi_{NS}(s', +; s, m) &= (-1)^{2m}\Phi_R(s', -; s, m) = \frac{e^{4\pi i s s'} + e^{-4\pi i s s'}}{4\sinh(\pi s')}\,,\\ \Phi_{NS}(s', -; s, m) &= (-1)^{2m}\Phi_R(s', +; s, m) = (-1)^{2m}\frac{e^{4\pi i s s'} + e^{-4\pi i s s'}}{4\cosh(\pi s')}\,,\end{aligned} \qquad (3.25)$$

and the corresponding reflection-invariant one-point functions on the disc

$$\langle \mathcal{V}^{NS+NS+}_{\frac{1}{2}+is', m', \bar{m}'}(p^\mu)\rangle_{s,m} = (-)^{2m}\langle \mathcal{V}^{R-R-}_{\frac{1}{2}+is', m'+\frac{1}{2}, \bar{m}'+\frac{1}{2}}(p^\mu)\rangle_{s,m}$$

[10] Because of different conventions these are A-type boundary states in [22] - see eq. (3.21) in that paper.

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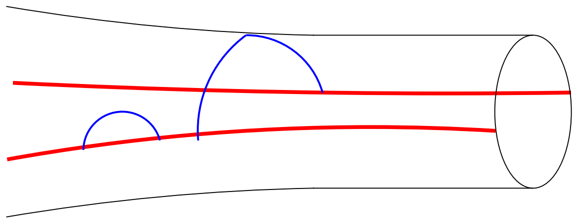

Figure 3: A D1-brane with two branches on the T-dual trumpet geometry. Open strings stretching on the same branch have integer windings whereas open strings stretching between different branches have half-integer windings. This configuration maps to a double-sheeted D2-brane on the cigar.

$$
\begin{aligned}
&= \delta^{(4)}(p^\mu)\delta_{m',\bar{m}'}\frac{\Gamma(1-2is')\Gamma(-2is')}{\Gamma(\frac{1}{2}-is'+m')\Gamma(\frac{1}{2}-is'-m')}\cos(4\pi s s')\,, \\
&\qquad m' \in \mathbb{Z}\,, \qquad\qquad (3.26)\\
\langle \mathcal{V}^{NS-NS-}_{\frac{1}{2}+is',m',\bar{m}'}(p^\mu)\rangle_{s,m} &= (-)^{2m}\langle \mathcal{V}^{R+R+}_{\frac{1}{2}+is',m'+\frac{1}{2},\bar{m}'+\frac{1}{2}}(p^\mu)\rangle_{s,m} \\
&= (-)^{2m}\delta^{(4)}(p^\mu)\delta_{m',\bar{m}'}\frac{\Gamma(1-2is')\Gamma(-2is')}{\Gamma(\frac{1}{2}-is'+m')\Gamma(\frac{1}{2}-is'-m')}\cos(4\pi s s')\,, \\
&\qquad m' \in \mathbb{Z}+\frac{1}{2}\,. \qquad\qquad (3.27)
\end{aligned}
$$

s is a non-negative real number and $m = 0, \frac{1}{2}$. We should emphasize that these boundary states are automatically consistent because they have been derived by T-duality from consistent branes of the $\mathcal{N} = 2$ Liouville theory.

Later in this section we will see that these branes contain open string states with both integer and half-integer momenta. In the asymptotic cylinder region, this implies that the class 2 boundary states appearing in (3.25) can be thought of as a superposition of two branes where the U(2) gauge symmetry is broken down to U(1) $\times$ U(1) by the presence of a Wilson line. An alternative but equivalent picture of the same effect is provided by the corresponding D1-brane on the T-dual trumpet. In the asymptotic cylinder this brane has two branches as well (see figure 3) and the open strings have integer or half-integer winding numbers depending on whether they stretch between the same or different branches. The angular separation of the two branches by an angle $\Delta\theta = \pi$ translates after T-duality to a non-trivial Wilson line between the two "sheets" of the cigar D2-brane.

Near the tip of the cigar, the geometry of the branes described by the class 2 boundary states depends crucialy on the parameter s in (3.25) which parametrizes the minimum distance of the branes from the tip. For $s > 0$ the two sheets are continuously connected (the geometry of the T-dual brane appears in figure 6); the worldvolume geometry can be depicted as a two-dimensional sheet coming from the asymptotic infinity up to a distance s from the tip, and then folding back to the asymptotic infinity with opposite orientation. As we send the modulus s to zero the radius of the angular direction at the turning point goes to zero as well. In this sense, for $s = 0$ we can think of the class 2 boundary states as describing two independent sheets fully covering the cigar. Further discussion on the geometry and the properties of these branes can be found in section 5.

On a technical level, one may be tempted to associate the two exponentials $e^{\pm 4\pi i s s'}$ in the wavefunctions (3.25) to the two more fundamental sheets that have different orien-

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tations. If we do that, we find that the spectrum of the resulting branes contains again both integer and half-integer momenta. This is not what we expect from decomposed one-sheeted D2-branes. Trying to further decompose these boundary states by separating different exponential contributions in the wavefunctions leads to boundary states that violate the Cardy consistency conditions with the class 1 brane. Hence, such decompositions do not appear to be admissible and they will not be discussed further in this paper.

Class 3. According to the general discussion of D-branes in $SL(2)/U(1)$, this class should contain boundary states with open strings in the *discrete* representations. In the present case, there are only two discrete representations (with $j = \frac{1}{2}, 1$) and they are both closely related to the continuous representation with $s = 0$. Hence, the application of the modular bootstrap does not lead to a genuinely new class of branes. It simply reproduces a class 2 boundary state with $s = 0$.

A different class of D2-branes (dubbed D2 cut branes in [26]) has been formulated for generic levels k in [16, 19]. In general, these branes have negative multiplicities in the open string channel and do not satisfy the Cardy consistency conditions. Recently, it was argued in [26] that this problem does not exist for integer levels k, because the dangerous discrete couplings in the closed string channel disappear. These branes are labeled by a single parameter $\sigma = \frac{\pi(2J-1)}{k}$, with $2J \in \mathbb{N}$ and $\frac{1}{2} < J < \frac{k+1}{2}$. For $k = 1$ there are no J's in this range. It is interesting to notice, however, that the special case $\sigma = \frac{\pi}{2}$, or $J = \frac{3}{4}$, reproduces the $s = 0$ class 2 boundary state of the previous paragraph.

3.2 B-type boundary states

The analysis of B-type boundary states is technically similar to that of the A-type boundary states appearing above and we will not repeat it here. There are a few differences, however, which should be pointed out. First, as we mentioned earlier, the Ramond part of the B-type Ishibashi states is projected out by the GSO projection.[11] Thus, all the B-type boundary states (with Neumann boundary conditions in the flat directions) will be non-BPS. A second important point is the absence of consistent B-type class 1 boundary states. This was argued for generic levels k (integers included) in [22]. Consequently, one is left with a set of class 2 boundary states in the NSNS sector only, which can be formulated as above (with a few appropriate modifications in the wavefunctions).

3.3 Cylinder amplitudes

In this subsection we compute the cylinder/annulus amplitudes of the above class 1 and class 2 boundary states. The modular transformation of these amplitudes from the closed string channel (parameter T) to the open (parameter $t = 1/T$) yields the explicit form of the spectral densities and the degeneracies of the open strings stretching between the various branes. We omit a detailed analysis of A-B and B-B overlaps, because they involve non-supersymmetric D-brane configurations that lie outside the immediate scope of this paper.

[11] Recall that we are considering the type-IIB superstring and D-branes that have Neumann boundary conditions in all four flat directions.

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Class 1 - class 1. By straightforward computation we find the following annulus amplitudes between class 1 boundary states:[12]

$$
{}_{NS}\langle A|e^{-\pi T H^c}|A\rangle_{NS} = \frac{1}{2}\left(\chi_I(it)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](it)}{\eta(it)^3} - \chi_I(it)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](it)}{\eta(it)^3}\right), \tag{3.28}
$$

$$
{}_{R}\langle A|e^{-\pi T H^c}|A\rangle_{R} = \frac{1}{2}\left(-\chi_I(it)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](it)}{\eta(it)^3} + \chi_I(it)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](it)}{\eta(it)^3}\right). \tag{3.29}
$$

The boundary state describing a BPS D3-brane is $|A\rangle = |A\rangle_{NS} + |A\rangle_R$, whereas that describing a D3-antibrane is $|\overline{A}\rangle = |A\rangle_{NS} - |A\rangle_R$. The self-overlaps of these boundary states are the same

$$
\begin{aligned}
\langle A|e^{-\pi T H^c}|A\rangle &= \langle \overline{A}|e^{-\pi T H^c}|\overline{A}\rangle \\
&= \frac{1}{2}\left(\chi_I(it)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](it)}{\eta(it)^3} - \chi_I(it)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](it)}{\eta(it)^3} - \right. \\
&\qquad \left. - \chi_I(it)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](it)}{\eta(it)^3} + \chi_I(it)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](it)}{\eta(it)^3}\right).
\end{aligned} \tag{3.30}
$$

As expected by supersymmetry both of them are vanishing. This can be demonstrated most easily in the closed string channel with the use of the vanishing character combinations $\Lambda_{\pm 1}(s;\tau)$ in (2.27), (2.28).

Class 2 - class 2. The class 2 boundary states $|A;s,m\rangle_{NS/R}$, defined in (3.25), exhibit the following amplitudes. In the NS-sector

$$
\begin{aligned}
&{}_{NS}\langle A;s_1,m_1|e^{-\pi T H^c}|A;s_2,m_2\rangle_{NS} = \\
&\quad = \int_0^\infty ds \sum_{m\in\mathbb{Z}_2}\left(\left(\rho_1(s;s_1|s_2) + (-1)^{2m_1+2m_2+m}\rho_2(s;s_1|s_2)\right)\times\right. \\
&\qquad\qquad \times \chi_c\left(s,\frac{m}{2};it\right)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](it)}{\eta(it)^3} - \Big(\rho_1(s;s_1|s_2) - \\
&\qquad\qquad\qquad - (-1)^{2m_1+2m_2+m}\rho_2(s;s_1|s_2)\Big)\times \\
&\qquad\qquad \left.\times \chi_c\left(s,\frac{m}{2};it\right)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](it)}{\eta(it)^3}\right),
\end{aligned} \tag{3.31}
$$

with spectral densities

$$
\rho_1(s;s_1|s_2) = 2\int_0^\infty ds' \frac{\cos(4\pi s' s_1)\cos(4\pi s' s_2)\cos(4\pi s s')}{\sinh(2\pi s')\tanh(\pi s')}, \tag{3.32}
$$

and

$$
\rho_2(s;s_1|s_2) = 2\int_0^\infty ds' \frac{\cos(4\pi s' s_1)\cos(4\pi s' s_2)\cos(4\pi s s')}{\sinh(2\pi s')\coth(\pi s')}. \tag{3.33}
$$

[12]In the rhs of the annulus amplitudes that appear in the ensuing, a factor of $\frac{1}{t}$ is omitted for simplicity.

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Similarly, in the R-sector

$$
\begin{aligned}
&{}_R\langle A; s_1, m_1| e^{-\pi T H^c} |A; s_2, m_2\rangle_R = \\
&= -\int_0^\infty ds \sum_{m\in\mathbb{Z}_2} \Bigg(\Big((-1)^{2m_1+2m_2+m} \rho_1(s; s_1|s_2) + \rho_2(s; s_1|s_2) \Big) \times \\
&\qquad \times \chi_c\left(s, \frac{m}{2}; it\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\theta[{}^0_1](it)}{\eta(it)^3} + \\
&\qquad + \Big((-1)^{2m_1+2m_2+m} \rho_1(s; s_1|s_2) - \rho_2(s; s_1|s_2) \Big) \times \\
&\qquad \chi_c\left(s, \frac{m}{2}; it\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\theta[{}^1_1](it)}{\eta(it)^3} \Bigg) .
\end{aligned} \tag{3.34}
$$

The total densities appearing in front of the continuous characters in the above amplitudes are $\rho_1(s; s_1|s_2) \pm \rho_2(s; s_1|s_2)$ depending on the precise values of m_1, m_2 and m. The spectral density $\rho_1(s; s_1|s_2)$ has an infrared divergence at $s' = 0$ associated to the infinite volume of the non-compact cigar geometry. As usual, this divergence can be regulated by subtracting the amplitude of a reference boundary state labeled by s^*. We will not specify a particular reference brane here.

In quantum theories with reflecting potentials there is a general relation between the density of continuous states and the appropriate reflection amplitudes (for a review of this argument see [55]). We can verify this relation explicitly in our case. Indeed, we obtain

$$
\rho_1(s; s_1|s_2) + \rho_2(s; s_1|s_2)\Big|_{\rm rel} = \frac{1}{2\pi i}\frac{\partial}{\partial s}\left(\log \frac{R(s, \frac{1}{2}|\pi(s_1+s_2))}{R(s, \frac{1}{2}|2\pi s^*)} + \log \frac{R(s, \frac{1}{2}|\pi(s_1-s_2))}{R(s, \frac{1}{2}|0)} \right), \tag{3.35}
$$

$$
\rho_1(s; s_1|s_2) - \rho_2(s; s_1|s_2)\Big|_{\rm rel} = \frac{1}{2\pi i}\frac{\partial}{\partial s}\left(\log \frac{R(s, 0|\pi(s_1-s_2))}{R(s, 0|0)} + \log \frac{R(s, 0|\pi(s_1+s_2))}{R(s, 0|2\pi s^*)} \right), \tag{3.36}
$$

with reflection amplitudes

$$
R(s, 0|r) = \frac{\Gamma_1^2(\frac{1}{2} - is)\Gamma_1(2is+1) S_1^{(0)}(s + \frac{r}{\pi})}{\Gamma_1^2(\frac{3}{2} + is)\Gamma_1(-2is+1) S_1^{(0)}(-s + \frac{r}{\pi})} \tag{3.37}
$$

for integer momenta, and

$$
R\left(s, \frac{1}{2}\middle| r\right) = \frac{\Gamma_1^2(\frac{1}{2} - is)\Gamma_1(2is+1) S_1^{(1)}(s + \frac{r}{\pi})}{\Gamma_1^2(\frac{3}{2} + is)\Gamma_1(-2is+1) S_1^{(1)}(-s + \frac{r}{\pi})} \tag{3.38}
$$

for half-integer momenta. The q-gamma functions $S_1^{(0)}(x)$ and $S_1^{(1)}(x)$ are defined as

$$
\log S_k^{(0)}(x) = i\int_0^\infty \frac{dt}{t}\left(\frac{\sin \frac{2tx}{k}}{2\sinh\frac{t}{k}\sinh t} - \frac{x}{t} \right), \tag{3.39}
$$

$$
\log S_k^{(1)}(x) = i\int_0^\infty \frac{dt}{t}\left(\frac{\cosh t \sin \frac{2tx}{k}}{2\sinh\frac{t}{k}\sinh t} - \frac{x}{t} \right). \tag{3.40}
$$

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The generalized gamma functions Γ_k can be found, for example in [16]. We do not present the explicit form of these functions here since they cancel out in the full eqs. (3.35) and (3.36) for the relative densities. Similar expressions for the spectral densities have been found in [16] and [19].

At this point we would like to make two comments. First, for a single brane, *i.e.* for an amplitude with $s_1 = s_2 = s$ and $m_1 = m_2$, the spectral density of modes with momentum m appears as a function of the reflection amplitude with quantum number $m + \frac{1}{2}$ mod 1. For instance, the density of open string modes with integer momentum m in the NS-sector is $\rho_1(s; s_1|s_1) + \rho_2(s; s_1|s_1)\Big|_{\text{rel}}$. In (3.35) we see that the corresponding reflection amplitude is $R(s, \frac{1}{2}|2\pi s_1)$. It would be interesting to understand this feature better. Secondly, with the current normalization of the class 2 branes (3.25) the expressions (3.35) and (3.36) agree with the general formula $\rho(s) = \frac{1}{2\pi i}\frac{\partial}{\partial s}\log\frac{R(s)}{R^*(s)}$. The current normalization of the class 2 branes can also be fixed independently by requiring that the class 1-class 1 and class 1-class 2 overlaps give the expected multiplicity of massless open string modes. Further arguments in favor of this normalization and the associated multiplicities will be given in section 5.

BPS boundary states can be formulated as before. They are given by the linear combinations

$$\begin{aligned} |A; s, m\rangle &= |A; s, m\rangle_{NS} + |A; s, m\rangle_R \\ |\overline{A; s, m}\rangle &= |A; s, m\rangle_{NS} - |A; s, m\rangle_R \end{aligned} \tag{3.41}$$

and they have vanishing self-overlaps as expected from supersymmetry.

Class 1 - class 2. We conclude this section with a brief survey of the cylinder amplitudes between class 1 and class 2 branes. The explicit form of these amplitudes is

$${}_{NS}\langle A|e^{-\pi T H^c}|A; s, m\rangle_{NS} = \frac{1}{2}\left(\chi_c(s, m; it)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](it)}{\eta(it)^3} - \chi_c\left(s, m+\frac{1}{2}; it\right)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](it)}{\eta(it)^3}\right), \tag{3.42}$$

$${}_{R}\langle A|e^{-\pi T H^c}|A; s, m\rangle_{R} = -\frac{1}{2}\left(\chi_c(s, m; it)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](it)}{\eta(it)^3} - \chi_c\left(s, m+\frac{1}{2}; it\right)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](it)}{\eta(it)^3}\right). \tag{3.43}$$

Supersymmetric D-brane configurations can be deduced from the vanishing amplitudes

$$\begin{aligned} \langle A|e^{-\pi T H^c}|A; s, 0\rangle &= \frac{1}{2}\Lambda_1(s; it) = 0\,, \\ \langle A|e^{-\pi T H^c}\left|\overline{A; s, \frac{1}{2}}\right\rangle &= \frac{1}{2}\Lambda_{-1}(s; it) = 0\,. \end{aligned} \tag{3.44}$$

3.4 A brief summary of the proposed D-branes

In the preceding analysis we considered D-branes in the four-dimensional non-critical type-IIB superstring theory (1.5) that have Neumann boundary conditions in the four flat directions, varying dimensionality in SL(2)/ U(1) and different BPS properties. D-branes in the type-IIA or type-IIB theory with lower dimensionality in $\mathbb{R}^{3,1}$ can be obtained easily by T-duality and will not be discussed here explicitly.

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More precisely, we found a D3-brane (denoted by the boundary state $|A\rangle$) and its anti-brane, both of which are separately BPS. The worldvolume of this brane is supported near the tip of the cigar. We also obtained D4- and D5-branes which are extended in the radial direction of the cigar. Both of these branes are labeled by a non-negative continuous real parameter s and an extra $\mathbb{Z}_2$ label $m = 0, \frac{1}{2}$. The B-type, class 2 D4-branes are non-BPS since they couple only to NSNS sector states. On the other hand, the D5-branes denoted by the boundary state $|A, s, m\rangle$ are BPS. Geometrically, the D5-branes cover the cigar partially or totally starting from the asymptotic circle at infinity and terminating at a finite distance $\rho_{\min} \sim s \geq 0$ from the tip. The analysis of the corresponding annulus amplitudes revealed that the D5-branes are double-sheeted, *i.e.* they have two branches in the T-dual trumpet geometry.

4. General properties of the BPS branes

The BPS D3- and D5-branes of the previous section are sources for the appropriate RR fields of the non-critical theory. In this section we want to elaborate on the nature of the corresponding RR couplings and the potential presence of dangerous non-dynamical RR tadpoles. In the process we also discuss the dictionary between branes in the non-critical superstring theory and branes in the corresponding NS5-brane configuration of [56, 57, 10].

As explained in section 2, from the six-dimensional point of view the type-IIB theory has RR fields coming from the R$-$R$-$ and R+R+ sectors. The massless RR potentials are

$$C_0\,,\ C_2^+\,,\ C_4 \tag{4.1}$$

and appear only in the R+R+ sector at the bottom of a continuous spectrum.

In the critical superstring, D3-branes couple electrically to the four-form potential C_4 through the standard WZ coupling

$$\int d^4x\, C_4\,. \tag{4.2}$$

In the present non-critical case, this statement is slightly obscured by the non-trivial profile of the D3-brane, which extends along the radial direction of the cigar but is mainly supported near the tip. Furthermore, the class 1 boundary conditions on the free fermions of the theory are Neumann in all directions; in particular, they are Neumann in both the radial and the angular directions of the cigar. In other words, one has to impose on the free fermions the same boundary conditions as in the case of the class 2 D5-branes, which we formulated with the use of the same Ishibashi states. In that sense, it is more appropriate to think of the class 1 D3-branes as small D5-branes localized near the tip of the cigar. Hence, in order to understand how they couple to RR fields it helps to understand first the corresponding couplings of the D5-branes.

In flat spacetime, D5-branes couple electrically to a six-form potential C_6. In the present non-critical case, six dimensions account for the full dimensionality of spacetime and the six-form is a non-dynamical field - the analog of the C_{10} potential in ten-dimensional flat spacetime, whose source is the D9-brane in type IIB. In ten dimensions a configuration of D9-branes with a non-vanishing C_{10} tadpole is a serious problem. Such tadpoles are usually

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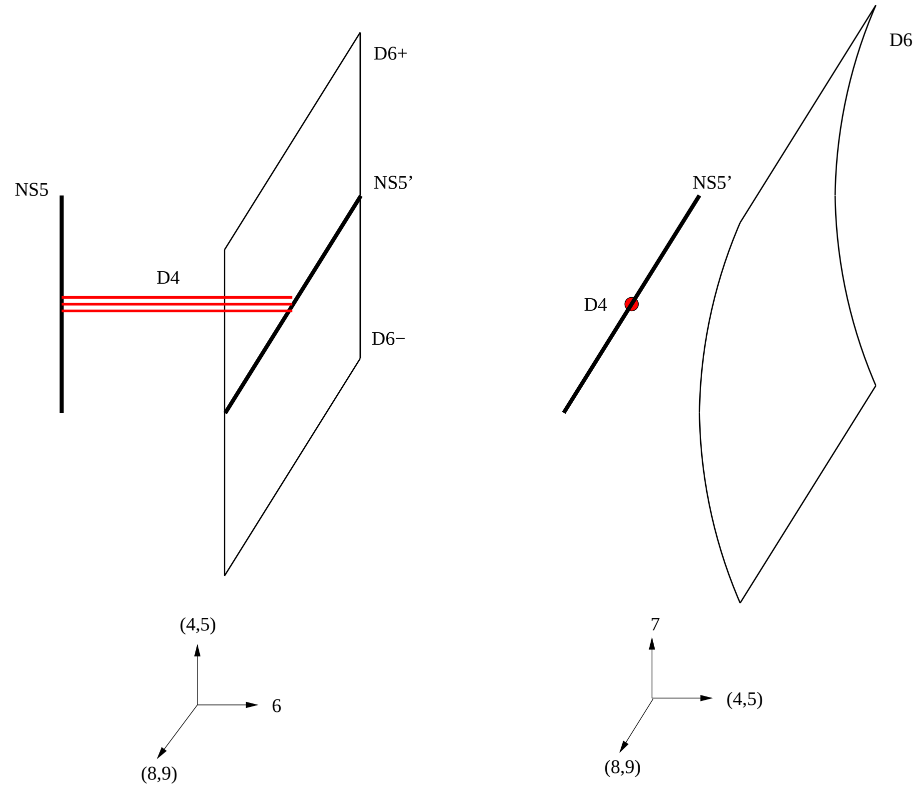

Figure 4: The NS5-brane configuration of figure 1 including D6-branes. On the left, the NS5'-brane is embedded inside a D6-brane extended in x^7. On the right, the D6-brane has been moved on the $(4,5)$ plane away from the origin and comes within a minimum distance from the NS5'-brane without intersecting it.

cancelled by introducing orientifold planes or the appropriate number of anti-D9-branes. Is there a similar C_6 tadpole from the D5 boundary states $|A;s,m\rangle$ in the non-critical case? We would like to argue that the answer to this question is negative. First of all, the boundary states $|A;s,m\rangle$ describe D5-branes with two sheets of opposite orientation. Asymptotically in the radial direction of the cigar, this configuration resembles a brane-antibrane pair and hence should have a vanishing C_6 charge. Despite this feature this system is supersymmetric and does not exhibit any open string tachyons. Secondly, the absence of any pathological non-dynamical tadpoles is expected to mesh nicely with the corresponding picture in the type-IIA NS5-brane configuration, which appears in figure 4. The correspondence with this configuration is another interesting aspect of the present discussion and we would to take a minute to summarize some of the relevant details.

In figure 4, the finite D4-branes suspended along the 6-direction between the NS5-branes

$$\begin{aligned} NS5: &\ (x^0,x^1,x^2,x^3,x^4,x^5)\,, \\ NS5': &\ (x^0,x^1,x^2,x^3,x^8,x^9) \end{aligned} \tag{4.3}$$

correspond to the class 1 D3-branes of the non-critical superstring setting. Accordingly,

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the type-IIA D6-branes

$$D6: \ (x^0, x^1, x^2, x^3, x^7, x^8, x^9) \tag{4.4}$$

have similar characteristics with the D5-branes $|A; s, 0\rangle$, $\overline{|A; s, \frac{1}{2}\rangle}$, which are T-dual to the D4-branes of figure 6 in the trumpet geometry.[13]

Both the D6-brane of figure 4 and the D4-branes of figure 6 come from the asymptotic infinity towards the throat and then return back. When the D6-branes of figure 4 approach the NS5'-brane they can intersect it at $x^4 = x^5 = x^7 = 0$ (see the figure on the left) or they can come within a minimum distance of the NS5'-brane at a locus of points with $x^4, x^5 \neq 0$, and $x^7 = 0$ (see the figure on the right). The special situation where the D6-branes meet the NS5'-brane at $x^7 = 0$ corresponds to the non-critical D5-branes with $s = 0$. In that case, the upper and lower sheets of the D6-brane correspond to the two separate sheets of the D5-brane. Clearly, we do not expect non-dynamical tadpoles in figure 4 and the same goes for the class 2 D5-branes in the non-critical superstring theory.

Nevertheless, we still observe that the D5-branes have a non-zero coupling to massless R+R+ potentials. What is the rank of these potentials and how do they couple to a six-dimensional worldvolume?

On the level of the effective spacetime action there are several ways that the dynamical RR potentials couple to the D5-branes of section 3. First of all, it is known [54] that D2-branes on the cigar can have a non-vanishing background gauge field strength F_2 on their worldvolume.[14] This implies that the spacefilling D5-branes can have WZ couplings of the form

$$\int d^6x \ e^{-\Phi} F_2 \wedge C_4 \,. \tag{4.5}$$

The presence of this coupling indicates that the class 2 branes of section 3 have an induced D3-brane charge and it would be interesting to understand its implications for the analysis of [11].

It is an open question whether there exist any non-trivial WZ couplings due to the curvature of the cigar. An obvious choice is a coupling of the form

$$\int d^6x \ e^{-\Phi} \mathrm{Tr}(R \wedge R) \wedge C_2^+ \,. \tag{4.6}$$

We are not aware of an explicit demonstration of such WZ couplings in the non-critical superstring case, but it would be useful to derive and verify their presence with a tree-level calculation on the disc. Analogous statements should apply also to the D3-brane boundary states, which are based on the same Ishibashi states as the D5-branes and therefore should couple to the C_4 RR potential in a similar fashion.

Finally, a potentially worrying aspect of having a D-brane setup with non-vanishing D3-brane flux is the following. A D3-brane in our six-dimensional non-critical setting is similar to a D7-brane in ten-dimensional flat space, which is pathological. The origin of

[13] We will say more about this correspondence in section 5 below. For example, the $|A; s, 0\rangle$ and $\overline{|A; s, \frac{1}{2}\rangle}$ branes exhibit some important differences.

[14] As we are about to see in the next section, there is no massless gauge field on the D5-branes, but there is a massless scalar which can be thought of as the fluctuation of a two-form field strength F_2.

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the pathology lies in the low co-dimension that does not allow the flux lines to decay appropriately fast in the asymptotic infinity. For a D7-brane in ten dimensions, the co-dimension is two and the solution of the Laplace equation in the two-dimensional transverse space is logarithmic suggesting that we cannot ignore the backreaction of the brane.

At first sight, the same conclusion would seem to hold for a D3-brane in our six-dimensional space. A more careful examination, however, shows that this is not the case. The two-dimensional Laplace equation on the axially-gauged cigar geometry of $\mathrm{SL}(2)_1/\mathrm{U}(1)$ takes the form [31]

$$\left[\frac{\partial^2}{\partial\rho^2} + \coth\rho\frac{\partial}{\partial\rho} + \coth^2\frac{\rho}{2}\frac{\partial^2}{\partial\theta^2}\right]T(\rho,\theta) = 0\,, \tag{4.7}$$

which becomes

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\theta^2}\right]T(\rho,\theta) = 0 \tag{4.8}$$

at the asymptotic region $\rho \to \infty$. For wavefunctions of the form $T(\rho,\theta) = f(\rho)e^{im\theta}$ this equation has two solutions for $f(\rho)$, one exponentially growing and another exponentially decaying. Hence the problem with the logarithmic divergence does not appear.

5. Four-dimensional gauge theories on D3-D5 systems

We are now in position to realize the main purpose of this paper, which is to obtain four-dimensional $\mathcal{N} = 1$ SQCD as the low-energy theory of the modes living on a configuration of D-branes in the four-dimensional non-critical superstring (1.5). $\mathcal{N} = 1$ SQCD is an $\mathrm{SU}(N_c)$ super-Yang-Mills theory with N_f flavour chiral superfields Q^i in the fundamental $\mathbf{N}_c$ of the gauge group and N_f flavour chiral superfields $\tilde{Q}_{\tilde{i}}$ in the anti-fundamental $\bar{\mathbf{N}}_c$ ($i, \tilde{i} = 1, \ldots, N_f$). For $N_f \leq 3N_c$ this theory is asymptotically free and has an infrared behaviour that depends crucially on N_c and N_f. In particular, for $N_f > N_c + 1$ it exhibits a very interesting electric-magnetic duality, known as Seiberg-duality [58], which exchanges the above electric description with a dual magnetic one that has different ultraviolet properties but the same infrared behaviour. The classical symmetries and moduli of $\mathcal{N} = 1$ SQCD will be discussed later in this section, where it will be examined which properties of the gauge theory can be realized directly in a D-brane setup in non-critical superstring theory.

5.1 The D-brane setup and the spectrum of open strings

The SYM part of $\mathcal{N} = 1$ SQCD can be realized on N_c D3-branes at the tip of the cigar. The spectrum of 3-3 strings can be deduced from the amplitude $\langle A|e^{-\pi T H^c}|A\rangle$ in section 3 and contains massless fields that belong in a $\mathcal{N} = 1$ vector supermultiplet. Indeed, the 3-3 open string spectrum comprises of a bosonic NS+ sector and a fermionic R− sector. The leading order expansion of the NS+ sector character gives two physical massless modes

$$\frac{1}{2}\left(\chi_I(it)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](it)}{\eta(it)^3} - \chi_I(it)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](it)}{\eta(it)^3}\right) \sim 2 + \mathcal{O}(q) \tag{5.1}$$

and the same result holds for the R− sector as well. This is the right multiplicity for the physical modes of a four-dimensional gauge field and the corresponding gauginos. Hence,

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putting N_c D3-branes on top of each other gives the full spectrum of pure $\mathrm{U}(N_c)$ super Yang-Mills.[15]

One can realize the chiral superfields Q^i and $\tilde{Q}_{\tilde{i}}$ with an extra set of N_f D5-branes. In the language of section 3 these should be A-type class 2 branes and the available boundary states are

$$|A; s, m\rangle\,, \qquad \overline{|A; s, m\rangle}\,, \qquad s \in \mathbb{R}_{\geq 0}\,, \qquad m = 0, \frac{1}{2}\,. \tag{5.2}$$

In the presence of D3-branes only the following subset of boundary states leads to supersymmetric configurations

$$|A; s, 0\rangle\,, \qquad \overline{\left|A; s, \frac{1}{2}\right\rangle}\,. \tag{5.3}$$

Since they are double-sheeted, we expect that N_f branes of this type will be sufficient in realizing the full matter content of $\mathcal{N} = 1$ SQCD, which includes an equal number of superfields in the fundamental and the anti-fundamental.

Indeed, these superfields will arise as the lowest level excitations of 3-5 strings. In section 3, we presented the annulus amplitudes

$$\begin{aligned}
\langle A|e^{-\pi T H^c}|A; s, 0\rangle &= \frac{1}{2}\Lambda_1(s; it) = 0\,, \\
\langle A|e^{-\pi T H^c}\overline{\left|A; s, \frac{1}{2}\right\rangle} &= \frac{1}{2}\Lambda_{-1}(s; it) = 0\,.
\end{aligned} \tag{5.4}$$

Massless excitations of 3-5 strings appear only in the character combination $\Lambda_{-1}(s; it)$ for the special case $s = 0$. Hence, from now on we concentrate on D5-branes represented by the boundary state $\overline{|A; s, \frac{1}{2}\rangle}$. For this choice 3-5 strings include at the lowest level an equal number of massless NS$-$ bosons and R$+$ fermions, which form *two* massless $\mathcal{N} = 1$ chiral multiplets. This can be seen directly from the character expansion

$$\begin{aligned}
\Lambda_{-1}(s; it) &= \left(\chi_c\left(s, \frac{1}{2}; it\right)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[{}^0_0](it)}{\eta(it)^3} + \chi_c\left(s, \frac{1}{2}; it\right)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[{}^0_1](it)}{\eta(it)^3}\right) - \\
&\quad - \left(\chi_c(s, 0; it)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[{}^1_0](it)}{\eta(it)^3} + \chi_c(s, 0; it)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[{}^1_1](it)}{\eta(it)^3}\right) \\
&= \left(4q^{s^2} + \mathcal{O}(q^{s^2+\frac{1}{2}})\right)_{NS-} - \left(4q^{s^2} + \mathcal{O}(q^{s^2+\frac{1}{2}})\right)_{R+}\,,
\end{aligned} \tag{5.5}$$

which is quoted here for arbitrary s. Moreover, using the character identities of appendix A we can rewrite $\Lambda_{-1}(0; \tau)$ in terms of discrete characters as

$$\begin{aligned}
\frac{1}{2}\Lambda_{-1}(0; \tau) &= \left(\chi_d\left(\frac{1}{2}, 0; \tau\right)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[{}^0_0](\tau)}{\eta(\tau)^3} + \chi_d\left(\frac{1}{2}, 0; \tau\right)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[{}^0_1](\tau)}{\eta(\tau)^3}\right) - \\
&\quad - \left(\chi_d\left(\frac{1}{2}, 1; \tau\right)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[{}^1_0](\tau)}{\eta(\tau)^3} + \chi_d\left(\frac{1}{2}, 1; \tau\right)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[{}^1_1](\tau)}{\eta(\tau)^3}\right).
\end{aligned} \tag{5.6}$$

[15] In the D-brane configurations of Hanany-Witten type the U(1) is frozen in the quantum theory and decouples [59]. Presumably the same happens also in our case. However, the quantum properties of the present configurations will not be discussed here, since they lie outside the immediate scope of this paper.

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It is natural to interpret the two lowest level contributions in (5.6) as the quark supermultiplets Q^i and $\tilde{Q}^\dagger_{\tilde{i}}$ respectively. Geometrically, these fields originate from 3-5 strings stretching between the D3-brane and different sheets of the D5-brane $|\overline{A;0,\frac{1}{2}}\rangle$. The superfields Q^i appear with momentum $n = \frac{1}{2}$ and transform in the fundamental representation $(\mathbf{N}_c, \mathbf{N}_f)$ of $\mathrm{U}(N_c) \times \mathrm{U}(N_f)$. The second set of chiral superfields $\tilde{Q}_{\tilde{i}}$ has the same momentum, transforms in the anti-fundamental $(\bar{\mathbf{N}}_c, \bar{\mathbf{N}}_f)$ and arises from the opposite orientation 5-3 strings.

The above picture is perfectly consistent with the one expected from the NS5-brane configuration in figure 4. In the situation depicted on the left of that figure the D6-brane splits into two pieces, which we call D6+ and D6−. Each of them corresponds to one of the sheets of the class 2 D5-brane $|\overline{A;0,\frac{1}{2}}\rangle$. Strings stretching between the D4-branes and D6+ are expected to give rise to the quark supermultiplets Q^i, whereas strings stretching between the D4-branes and D6− are expected to give rise to the quark supermultiplets $\tilde{Q}^\dagger_{\tilde{i}}$ [60, 15].

Consequently, in what follows we consider a setup of N_c D3-branes and N_f D5-branes described respectively by the boundary states $|A\rangle$ and $|\overline{A;0,\frac{1}{2}}\rangle$ and we argue that they realize the electric description of $\mathcal{N} = 1$ SQCD. The rôle of D5-branes with $s > 0$, will be clarified shortly. Note that the other class of D5-branes represented by the boundary state $|A; s, 0\rangle$ gives massive 3-5 spectra in the NS+, R− sectors and does not appear to play a rôle, when we try to engineer $\mathcal{N} = 1$ SQCD.

So far we have discussed the spectrum of 3-3 and 3-5 strings. Now we turn to the spectrum of 5-5 strings. This can be read off the annulus amplitude

$$\left\langle \overline{A;0,\frac{1}{2}} \middle| e^{-\pi T H^c} \middle| \overline{A;0,\frac{1}{2}} \right\rangle = \int_0^\infty ds' \Big[(\rho_1(s';0|0) + \rho_2(s';0|0))\Lambda_1(s';it) + \\ + (\rho_1(s';0|0) - \rho_2(s';0|0))\Lambda_{-1}(s';it) \Big] \,, \qquad (5.7)$$

where ρ_1, ρ_2 are the spectral densities of eqs. (3.35), (3.36). The most notable characteristics of this spectrum are the following. First, it does not exhibit any massless vector multiplets, which would correspond to massless gauge fields on the D5-branes. Vector multiplets appear in the NS+ and R− sectors, which are captured by the $\Lambda_1(s;\tau)$ character. There are no massless contributions to this character for any value of s. Although the existence of a massive vector seems strange at first sight, it is a natural characteristic of linear dilaton backgrounds.

The second notable characteristic of the spectrum (5.7) is a massless chiral multiplet $M_i^{\tilde{j}}$ in the bifundamental of $\mathrm{U}(N_f) \times \mathrm{U}(N_f)$ with quantum numbers $s = 0$, $m = 1/2$. This mode has a natural superpotential coupling to the quarks Q^i, $\tilde{Q}_{\tilde{j}}$

$$W_M = \mathrm{Tr}\, M_i^{\tilde{j}} Q^i \tilde{Q}_{\tilde{j}} \,, \qquad (5.8)$$

which can be deduced from the respective three-string tree-level interaction. Notice that a similar coupling appears in the magnetic description of SQCD for the elementary magnetic mesons. Hence, one may wonder whether we are really discussing the magnetic description

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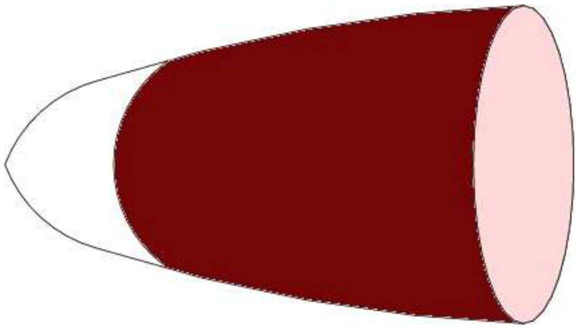

Figure 5: The geometric picture of a cigar D2-brane corresponding to a class 2 boundary state. It covers the cigar partially up to a minimum distance s from the tip.

of SQCD and if we should interpret the massless multiplets $M_i^{\tilde{j}}$ as the magnetic mesons of that description. However, the fact that the multiplets $M_i^{\tilde{j}}$ appear at the bottom of a continuous spectrum with arbitrary radial momentum in the cigar direction indicates that they do not constitute propagating UV degrees of freedom in the D3-brane gauge theory. Instead, they should be regarded as parameters in this gauge theory. The precise meaning of these parameters in the electric description of SQCD is the following.

The superpotential coupling (5.8) implies that vacuum expectation values (vev's) of the $M_i^{\tilde{j}}$ operators give masses to the quarks Q, $\tilde{Q}$ and generate (a subset of) the usual mass deformations of $\mathcal{N}=1$ SQCD. These deformations have a clear geometric meaning in our setup that can be understood by considering more closely the worldvolume theory of the flavor branes. We can see directly from equations (5.4) and (5.5) that turning on the mass parameter $M_i^{\tilde{i}}$ for the single ith D5-brane corresponds to shifting the modulus s of the class 2 branes by an amount s_i proportional to $|M_i^{\tilde{i}}|$.[16] Hence, by turning on this deformation we expect to get the class 2 boundary state of a D5-brane that wraps the cigar and extends from the asymptotic infinity up to a distance s_i from the tip (see figure 5). Notice that in this process the two sheets of a single flavour brane cannot move independently and we can only obtain the diagonal vev's

$$M_i^{\tilde{j}} = m_i \delta_i^{\tilde{j}} \tag{5.9}$$

(no summation implied). Each vev m_i is in one-to-one correspondence with the single modulus s_i of the class 2 D5-branes

$$\left|\overline{A; s_i, \frac{1}{2}}\right\rangle . \tag{5.10}$$

5.2 Symmetries and moduli

At this point, we want to make a few general comments about the classical symmetries and moduli of $\mathcal{N}=1$ SQCD[17] and see if and how they can be realized geometrically in

[16] The existence of this mass deformation is another reason to expect two chiral multiplets in the spectrum of 3-5 strings and substantiates the validity of the normalization of the class 1 and class 2 boundary states in section 3. A single chiral multiplet cannot give rise to a holomorphic gauge-invariant mass deformation.

[17] The quantum moduli space is not accessible to our tree-level classical (type-II) string theory description.

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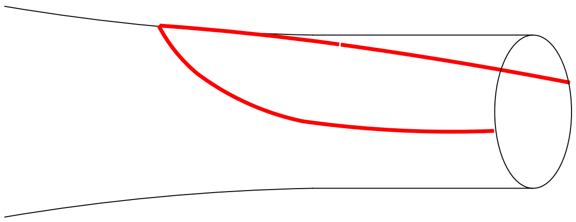

Figure 6: A D1-brane on the T-dual trumpet. The D1-brane comes from the asymptotic infinity, curves and then turns back to the asymptotic infinity at the diametrically opposite point.

the D-brane configurations of this paper. In the absence of a superpotential, the classical symmetry of the theory is

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_a \times \mathrm{U}(1)_x \,. \tag{5.11}$$

The two $\mathrm{SU}(N_f)$ factors rotate the chiral multiplets Q^i, $\tilde{Q}_{\tilde{j}}$. $\mathrm{U}(1)_B$ is a vector-like baryon symmetry, which assigns charge $+1$ (-1) to Q $(\tilde{Q})$. $\mathrm{U}(1)_a$ and $\mathrm{U}(1)_x$ are R-symmetries under which the gaugino has charge one and the quarks Q, $\tilde{Q}$ have charge 1 or 0. Quantum mechanically only one combination of the two R-symmetries is anomaly free.

The vector $\mathrm{SU}(N_f)$ global symmetry is present in any configuration with the same parameters s_i for all flavor branes. The appearance of a second axial $\mathrm{SU}(N_f)$, when all the matter multiplets are massless, can be seen more easily in the T-dual trumpet geometry. The mass deformed theory involves D5-branes with parameters $s > 0$, which look like the D1-branes of figure 6 in the T-dual trumpet. The two D1-branches are connected to each other and therefore exhibit a single (vector) $\mathrm{SU}(N_f)$ symmetry. For $s = 0$ however, the two branches are disconnected and go straight into the strong coupling singularity. Then we can associate an $\mathrm{SU}(N_f)$ symmetry to each one of the two independent branches, leading to an enhancement of the flavour symmetry to $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$. This reproduces exactly the field theory result (5.11). Similar statements in the context of the NS5-brane configuration in figure 4 can be found in [61, 60, 10].

The closed string theory on the cigar has three conserved U(1) currents [44]. Two of them are the chiral and anti-chiral $\mathcal{N} = 2$ currents $J_{\mathcal{N}=2}$ and $\bar{J}_{\mathcal{N}=2}$, and the third is the non-chiral current associated to the momentum in the angular direction of the cigar. Only the last current commutes with the BRST symmetry and constitutes a physical current of the non-critical superstring theory.[18]

Open string theory on the previously discussed D3-D5 configuration exhibits the following three U(1) symmetries. The first, which will be called $\mathrm{U}(1)_m$ (with charge Q_m) is the open string version of the U(1) angular momentum symmetry of the cigar. The other two, which will be called $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$ are the U(1) global symmetries, which are part of the $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ global flavor symmetry group. We will denote the corresponding charges as Q_L and Q_R. It is useful to define the following linear combinations of these

[18]We would like to thank S. Ashok, S. Murthy and J. Troost for pointing this out and for correcting an erroneous statement in the first version of this paper.

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charges

$$
\begin{aligned}
Q_\pm &= \frac{1}{2}(Q_L \pm Q_R)\,, \\
Q_x &= \frac{4}{3}(Q_m + Q_-)\,, \\
Q_a &= \frac{4}{3}\left(Q_m - \frac{1}{2}Q_-\right), \\
Q_B &= -2Q_+\,.
\end{aligned} \tag{5.12}
$$

The three charges Q_B, Q_a and Q_x are in one-to-one correspondence with the SQCD U(1) symmetries $\mathrm{U}(1)_B, \mathrm{U}(1)_a$ and $\mathrm{U}(1)_x$. Indeed, Q_B is the charge of a global U(1) symmetry and Q_x, Q_a, both of which involve the $\mathrm{U}(1)_m$ charge, are the charges of two $\mathrm{U}(1)_R$ symmetries.

One can check that with this identification the U(1) charge assignments work as expected from SQCD. The chiral superfield M has $\mathrm{U}(1)_B \times \mathrm{U}(1)_a \times \mathrm{U}(1)_x$ charges $(0, 0, 2)$ and the quark superfields Q and $\tilde{Q}$ have respectively $(1, 1, 0)$ and $(-1, 1, 0)$.

The geometric interpretation of the $\mathrm{U}(1)_R$ symmetries $\mathrm{U}(1)_a$ and $\mathrm{U}(1)_x$ is most clear in the T-dual trumpet background of figure 6. In that case, the momentum symmetry $\mathrm{U}(1)_m$ becomes a winding symmetry $\mathrm{U}(1)_w$. The charges Q_a and Q_x measure respectively the winding of open strings on the lower and upper half of the asymptotic cylinder in figure 6.[19] For D4-branes with $s > 0$ Q_a is conserved, but Q_x is not. Pictorially, strings winding around the bottom half of the asymptotic cylinder cannot unwind by reaching the turning point of the brane, but strings winding around the upper half can.[20] This fits nicely with the fact that the superfield M is charged under $\mathrm{U}(1)_x$, but uncharged under $\mathrm{U}(1)_a$. As a result, non-zero vev's of M break $\mathrm{U}(1)_x$ explicitly, but they preserve $\mathrm{U}(1)_a$.

We would like to finish with a few comments on the classical moduli space of $\mathcal{N} = 1$ SQCD. It is well-known that the dimensionality of this space depends crucially on the number of colours and flavours N_c and N_f respectively. For $N_f < N_c$ the moduli space is N_f^2 dimensional and can be labeled by the gauge invariant meson fields $Q^i \tilde{Q}_{\tilde{j}}$. By giving non-zero vev's to the massless quarks Q, $\tilde{Q}$ one can Higgs the gauge group down to $\mathrm{SU}(N_c - N_f)$. For $N_f \geq N_c$ new gauge invariant baryon fields appear and the dimension of the moduli space becomes $2N_cN_f - N_c^2$. The gauge group can now be broken completely by the Higgs mechanism.

In the brane description of [57], Higgsing corresponds to splitting fourbranes on sixbranes in the presence of NS5-branes according to the s-rule. In our setting, Higgsing corresponds to a marginal deformation of the open string theory living on our D3-D5 setup, but this deformation does not appear to have an obvious geometric meaning. If

[19] Strictly speaking, this is true for the linear combinations

$$
Q'_a = Q_m - \frac{1}{2}Q_-\,, \qquad Q'_x = 2\left(Q_m + \frac{1}{2}Q_-\right).
$$

[20] We would like to thank S. Murthy, who suggested this picture to us.

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Higgsing can be described geometrically in our setup, it would imply a non-trivial statement involving the D3-branes at the tip of the cigar. Obviously, it would be extremely interesting to understand this point better. Among other things, this could be useful for a microscopic derivation of the "phenomenological" s-rule of D-brane dynamics in the vicinity of NS5-branes. We hope to return to this interesting issue in future work.

6. Future prospects

In this paper we studied several aspects of D-brane dynamics in a specific four-dimensional non-critical superstring theory, which involves the $\mathcal{N} = 2$ Kazama-Suzuki model for $\mathrm{SL}(2)/\mathrm{U}(1)$ at level 1. D-branes in this theory were treated with exact boundary conformal field theory methods building on previous work on the $\mathcal{N} = 2$ Liouville theory and $\mathcal{N} = 2$ Kazama-Suzuki model with boundary [16 − 22]. A similar analysis for the more general case (1.1) can be performed with analogous techniques and it will be useful for a better understanding of D-brane dynamics in closely related situations involving non-critical superstring theory, string theory in the vicinity of Calabi-Yau singularities, and the near-horizon geometry of NS5-branes. In general, this study is expected to yield interesting information about gauge theories and LSTs. Related work in this direction has appeared recently in [26].

Our primary goal in this paper was to understand some of the key features of the general story by studying a specific example that realizes $\mathcal{N} = 1$ SQCD. There are several aspects of our analysis that deserve further study. For example, it would be very interesting to see if we can obtain the dual magnetic description of SQCD using D-branes in the non-critical superstring (1.5). This seems difficult to achieve solely with the D-branes presented in section 2. On the other hand, the general analysis of D-branes in the background of NS5-branes à la Hanany-Witten suggests that this should be possible. If so, can we also understand Seiberg duality as a classical statement of the corresponding D-brane configurations? Within the framework of NS5-brane setups [57, 10], or within its T-dual involving Calabi-Yau singularities [62], there are convincing arguments that demonstrate Seiberg duality in this way.

Another interesting question is whether the Higgs moduli of $\mathcal{N} = 1$ SQCD have a clear geometric meaning in terms of D-brane configurations in the non-critical superstring description. This would be a non-trivial statement involving the D3-branes at the tip of the cigar and may also lead to a microscopic derivation or at least further insight on the "phenomenological" s-rule of D-brane dynamics in the background of NS5-branes.

Finally, it would be extremely interesting to see whether we can obtain a better grasp of a generalized AdS/CFT correspondence within non-critical superstring theory along the lines of [11]. This would open up the road for a direct analysis of the strong coupling dynamics of the class of gauge theories that can be realized in non-critical superstring theory and the corresponding NS5-brane configurations. Clearly, one of the major tasks is to determine the backreaction of the D3- and D5-branes on the cigar geometry. A first step in this direction, using supergravity methods, has been taken in previous work [11] by Klebanov and Maldacena. They found a highly curved supergravity solution, which is

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relevant for $\mathcal{N}=1$ SQCD at the conformal window. A better understanding of this solution, *e.g.* in relation to its stability and Seiberg duality, can perhaps be obtained using the results presented here. For example, calculating the one-point function of massless closed string fields on the disc and their profile in the asymptotic infinity is a first exercise that can be done in a straightforward way using the results of this paper [63]. Of course, in order to proceed further one would have to compute and resum an infinite set of contributions coming from higher open string loops (see [64] for a similar analysis in the critical case). Also, going beyond supergravity is bound to bring in the complications due to RR fields. It would be interesting to see how far one can go and how useful it is to think about AdS/CFT within the setting of non-critical superstring theory.

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A. Useful formulae

A.1 Useful identities

For quick reference, we quote here a few identities involving the characters of discrete representations. First of all, one can show that the continuous characters for $s=0$ can be written as

$$\chi_c\left(s=0, \frac{a+1}{2}; \tau, z\right) \begin{bmatrix} a \\ b \end{bmatrix} = \chi_d\left(\frac{1}{2}, \frac{a}{2}; \tau, z\right) \begin{bmatrix} a \\ b \end{bmatrix} + (-)^b \chi_d\left(1, \frac{a}{2}; \tau, z\right) \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A.1}$$

With the use of the identity

$$\chi_d\left(1, \frac{a}{2}; \tau, -z\right) \begin{bmatrix} a \\ b \end{bmatrix} = (-)^{b+ab} \chi_d\left(\frac{1}{2}, \frac{a}{2}; \tau, z\right) \begin{bmatrix} a \\ b \end{bmatrix} \tag{A.2}$$

we can also write eq. (A.1) as

$$\chi_c\left(s=0, \frac{a+1}{2}; \tau, z\right) \begin{bmatrix} a \\ b \end{bmatrix} = (1+(-)^{ab})\ \chi_d\left(\frac{1}{2}, \frac{a}{2}; \tau, z\right) \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A.3}$$

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In the main text we also define the vanishing character combinations

$$\Lambda_1(s;\tau) = \left(\chi_c(s,0;\tau,0)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau,0)}{\eta(\tau)^3} - \chi_c(s,0;\tau,0)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](\tau,0)}{\eta(\tau)^3}\right) - $$
$$-\left(\chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau,0)}{\eta(\tau)^3} - \chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau,0)}{\eta(\tau)^3}\right) \equiv 0\,, \quad \text{(A.4)}$$

$$\Lambda_{-1}(s;\tau) = \left(\chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau,0)}{\eta(\tau)^3} + \chi_c(s,\frac{1}{2};\tau,0)\begin{bmatrix}0\\1\end{bmatrix}\frac{\theta[^0_1](\tau,0)}{\eta(\tau)^3}\right)$$
$$-\left(\chi_c(s,0;\tau,0)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau,0)}{\eta(\tau)^3} + \chi_c(s,0;\tau,0)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau,0)}{\eta(\tau)^3}\right) \equiv 0\,. \quad \text{(A.5)}$$

Using (A.1) and then (A.2) we can recast $\Lambda_{-1}(0;\tau)$ into the form

$$\begin{aligned}\Lambda_{-1}(0;\tau) = &\left\{\left(\chi_d(\frac{1}{2},0;\tau)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau)}{\eta(\tau)^3} + \chi_d(\frac{1}{2},0;\tau)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_1](\tau)}{\eta(\tau)^3}\right) - \right.\\
&\left. - \left(\chi_d(\frac{1}{2},1;\tau)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau)}{\eta(\tau)^3} + \chi_d(\frac{1}{2},1;\tau)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau)}{\eta(\tau)^3}\right)\right\} + \\
&+\left\{\left(\chi_d(1,0;\tau)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau)}{\eta(\tau)^3} - \chi_d(1,0;\tau)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_1](\tau)}{\eta(\tau)^3}\right) - \right.\\
&\left. - \left(\chi_d(1,1;\tau)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau)}{\eta(\tau)^3} - \chi_d(1,1;\tau)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau)}{\eta(\tau)^3}\right)\right\}\\
= 2&\left\{\left(\chi_d(1,0;\tau)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_0](\tau)}{\eta(\tau)^3} - \chi_d(1,0;\tau)\begin{bmatrix}0\\0\end{bmatrix}\frac{\theta[^0_1](\tau)}{\eta(\tau)^3}\right) - \right.\\
&\left. - \left(\chi_d(1,1;\tau)\begin{bmatrix}1\\0\end{bmatrix}\frac{\theta[^1_0](\tau)}{\eta(\tau)^3} - \chi_d(1,1;\tau)\begin{bmatrix}1\\1\end{bmatrix}\frac{\theta[^1_1](\tau)}{\eta(\tau)^3}\right)\right\}. \qquad \text{(A.6)}\end{aligned}$$

A.2 $\mathcal{S}$-modular transformation properties of the extended characters

Under the modular transformation $\mathcal{S}:\ \tau \to -\frac{1}{\tau}$ the extended characters presented in the main text transform in the following way (see for example [47]):

$$\begin{aligned}\chi_c\left(s,m;-\frac{1}{\tau},\frac{z}{\tau}\right)\begin{bmatrix}a\\b\end{bmatrix} &= 2(-i)^{ab}e^{3\pi i z^2/\tau}\sum_{m'\in\mathbb{Z}_2}e^{-2\pi i m m'}\int_0^\infty ds'\cos(4\pi s s')\times\\
&\quad\times\chi_c\left(s',\frac{m'}{2};\tau,z\right)\begin{bmatrix}b\\a\end{bmatrix}, \qquad \text{(A.7)}\\
\chi_d\left(j,\frac{a}{2};-\frac{1}{\tau},\frac{z}{\tau}\right)\begin{bmatrix}a\\b\end{bmatrix} &= (-i)^{ab}e^{3\pi i z^2/\tau}(-1)^{2bj}\times\\
&\quad\times\left[\int_0^\infty ds(-)^b\left\{\chi_c(s,0;\tau,z)\begin{bmatrix}b\\a\end{bmatrix} - (-)^a\chi_c\left(s,\frac{1}{2};\tau,z\right)\begin{bmatrix}b\\a\end{bmatrix}\right\} + \right.\\
&\quad + \frac{i}{2}(-)^{2j}(-)^{ab}\left\{(-)^a\chi_d\left(\frac{1}{2},\frac{b}{2};\tau,z\right)\begin{bmatrix}b\\a\end{bmatrix} - \right.\\
&\quad\left.\left. - \chi_d\left(1,\frac{b}{2};\tau,z\right)\begin{bmatrix}b\\a\end{bmatrix}\right\}\right]. \qquad \text{(A.8)}\end{aligned}$$

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Using (A.2) this modular identity can be recast into a simpler form

$$\begin{aligned}\chi_d\left(j,\frac{a}{2};-\frac{1}{\tau},\frac{z}{\tau}\right)\begin{bmatrix}a\\b\end{bmatrix} &= (-i)^{ab}e^{3\pi iz^2/\tau}(-1)^{2bj}\times\\ &\times\int_0^\infty ds(-)^b\left\{\chi_c(s,0;\tau,z)\begin{bmatrix}b\\a\end{bmatrix}-(-)^a\chi_c\left(s,\frac{1}{2};\tau,z\right)\begin{bmatrix}b\\a\end{bmatrix}\right\}-\\ &-i\delta_{ab,1}(-)^{2j}(-)^a\chi_d\left(\frac{1}{2},\frac{b}{2};\tau,z\right)\begin{bmatrix}b\\a\end{bmatrix}.\end{aligned}\tag{A.9}$$

Finally, for the identity characters we have

$$\chi_I\left(-\frac{1}{\tau},\frac{z}{\tau}\right)\begin{bmatrix}a\\0\end{bmatrix} = 2(-i)^{ab}e^{3\pi iz^2/\tau}\int_0^\infty ds\,\sinh(2\pi s)\times\tag{A.10}$$

$$\times\left\{\tanh(\pi s)\chi_c(s,0;\tau,z)\begin{bmatrix}0\\a\end{bmatrix}+(-)^a\coth(\pi s)\chi_c\left(s,\frac{1}{2};\tau,z\right)\begin{bmatrix}0\\a\end{bmatrix}\right\},$$

$$\chi_I\left(-\frac{1}{\tau},\frac{z}{\tau}\right)\begin{bmatrix}a\\1\end{bmatrix} = 2(-i)^{ab}e^{3\pi iz^2/\tau}\int_0^\infty ds\sinh(2\pi s)\times\tag{A.11}$$

$$\times\left\{\coth(\pi s)\chi_c(s,0;\tau,z)\begin{bmatrix}1\\a\end{bmatrix}+(-)^a\tanh(\pi s)\chi_c\left(s,\frac{1}{2};\tau,z\right)\begin{bmatrix}1\\a\end{bmatrix}\right\}.$$

A.3 $\mathcal{S}$-modular transformation properties of classical θ-functions

The standard definition of theta-functions is

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(\tau,z) = (-i)^{ab}\sum_{n=-\in\infty}^{\infty}(-)^{bn}q^{(n-a/2)^2/2}z^{n-a/2}.\tag{A.12}$$

Under the transformation $\mathcal{S}:\tau\to-\frac{1}{\tau}$ these characters transform as

$$\theta\begin{bmatrix}a\\b\end{bmatrix}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = (-i)^{ab}(-i\tau)^{1/2}e^{\pi iz^2/\tau}\theta\begin{bmatrix}b\\a\end{bmatrix}(\tau,z).\tag{A.13}$$

The Dedekind eta function is

$$\eta(\tau) = q^{1/24}\prod_{m=1}^{\infty}(1-q^m)\tag{A.14}$$

and transforms in the following way

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau).\tag{A.15}$$

B. Chiral GSO projection and the type-II torus partition sum

In this appendix we review the chiral GSO projection that leads to the non-critical superstring partition sum (2.25). We start by writing down the four-dimensional spin fields

$$S_{s_0,s_1} = e^{i(s_0H_0+s_1H_1)},\tag{B.1}$$

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where H_0, H_1 are the bosonized spacetime fermions and $s_0, s_1 = \pm\frac{1}{2}$. It is also useful to bosonize the total $\mathcal{N} = 2$ current with a canonically normalized boson Y so that

$$J_{\mathcal{N}=2} = i\sqrt{\hat{c}}\partial Y = i\sqrt{3}\partial Y\,. \tag{B.2}$$

We focus only on the case of interest $\hat{c} = 3 \Leftrightarrow k = 1$.

The type-II non-critical superstring has two sets of spacetime supercharges [2, 3]. One set originates from left-moving fields on the worldsheet and the other from right-moving fields. The spacetime supercharges coming from left-moving fields read

$$Q^{+}_{\frac{1}{2},\frac{1}{2}} = \oint dz\, e^{\frac{1}{2}(-\varphi+i\sqrt{3}Y)} S_{\frac{1}{2},\frac{1}{2}}\,, \qquad Q^{+}_{-\frac{1}{2},-\frac{1}{2}} = \oint dz\, e^{\frac{1}{2}(-\varphi+i\sqrt{3}Y)} S_{-\frac{1}{2},-\frac{1}{2}} \tag{B.3}$$

$$Q^{-}_{\frac{1}{2},-\frac{1}{2}} = \oint dz\, e^{\frac{1}{2}(-\varphi-i\sqrt{3}Y)} S_{\frac{1}{2},-\frac{1}{2}}\,, \qquad Q^{-}_{\frac{1}{2},-\frac{1}{2}} = \oint dz\, e^{\frac{1}{2}(-\varphi-i\sqrt{3}Y)} S_{-\frac{1}{2},\frac{1}{2}} \tag{B.4}$$

where φ bosonizes the superghost β,γ system. These supercharges are components of a six-dimensional spinor in the $\mathbf{4}$ of SO(5, 1), which can be decomposed as follows

$$\mathbf{4} \to \mathbf{2}_1 \oplus \bar{\mathbf{2}}_{-1} \tag{B.5}$$

under the decomposition $\mathrm{SO}(5,1) \to \mathrm{SO}(3,1)\times\mathrm{SO}(2)$. Hence, in four dimensions we obtain a Majorana spinor in the $\mathbf{2} \oplus \bar{\mathbf{2}}$ of SO(3, 1) yielding $N = 1$ spacetime supersymmetry. A similar set of spinors will arise from right-moving fields. More precisely, for the right-movers we have the option of choosing either the $\mathbf{4}$ or the $\mathbf{4}'$ corresponding to type-IIB or type-IIA non-critical superstring theory respectively. In four dimensions, both choices result in a four-dimensional Majorana spinor $\mathbf{2} \oplus \bar{\mathbf{2}}$, since

$$\mathbf{4}' \to \mathbf{2}_{-1} \oplus \bar{\mathbf{2}}_1\,. \tag{B.6}$$

The overall counting of supercharges yields a theory with $\mathcal{N} = 2$ supersymmetry in four dimensions. This meshes nicely with the fact that this non-critical string theory describes holographically a four-dimensional LST on a configuration of tilted NS5-branes or string theory near a conifold singularity, both of which preserve 1/4 of the ten-dimensional type-II supersymmetry.

On the level of vertex operators the GSO projection requires locality of all vertex operators with respect to the supercharges. For a vertex operator of the form

$$\exp\Bigg(\Big(-1+\frac{a}{2}\Big)\varphi + is_0H_0 + is_1H_1 + iQ_a\Big(\frac{Y}{\sqrt{3}}\Big)\Bigg) \tag{B.7}$$

this requirement yields the following integrality condition

$$J_{\mathrm{GSO}} = -1 + \frac{a}{2} + (s_0+s_1) + Q_a \,\in\, 2\mathbb{Z}\,. \tag{B.8}$$

$a = 0$ in the NS-sector and 1 in the R-sector. For $\mathcal{N} = 2$ primaries[21] the total $\mathrm{U}(1)_R$ charge reads

$$Q_a = 2\Big(m+\frac{a}{2}\Big) + \frac{a}{2}\,, \tag{B.9}$$

[21] For simplicity, we concentrate here only on the continuous representations. The discrete representations can be treated in the same way.

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where m is the J^3 charge of the corresponding bosonic $\mathrm{SL}(2)/\mathrm{U}(1)$ representation. The two a-dependent shifts in Q_a appear, because $J_{\mathcal{N}=2} = \psi^+\psi^- + \frac{2}{k}\mathcal{J}^3$ and $\mathcal{J}^3 = J^3 + \psi^+\psi^-$ is the global $\mathrm{U}(1)$ charge that we gauge in the supersymmetric $\mathrm{SL}(2)/\mathrm{U}(1)$. Sometimes, it is convenient to denote the eigenvalue of $\mathcal{J}^3$ by a separate parameter $m_t = m + a/2$. Then, we can write $J_{\mathrm{GSO}} = F + 2m_t$, where $F = -1 + a/2 + s_0 + s_1 + a/2$ is the total fermion number (including the superghost contribution).

In order to obtain a GSO invariant partition function we insert the projectors

$$\frac{1}{2}(1 + (-1)^{J_{\mathrm{GSO}}})\,, \qquad \frac{1}{2}(1 + (-1)^{\bar{J}_{\mathrm{GSO}}}) \tag{B.10}$$

inside the trace over the full Hilbert space $\mathcal{H}$ of the theory. This includes the 3+1-dimensional flat part, the supersymmetric coset and the ghosts. Hence,

$$Z_{\mathrm{II}} = \mathrm{Tr}_{\mathcal{H}}\left(\frac{1 + (-1)^{J_{\mathrm{GSO}}}}{2}\frac{1 + (-1)^{\bar{J}_{\mathrm{GSO}}}}{2} q^{L_0}\bar{q}^{\bar{L}_0}\right). \tag{B.11}$$

As usual, the contribution of two of the bosonic (fermionic) degrees of freedom is cancelled by the contribution of the ghosts (superghosts) and the trace ends up summing over the two transverse flat directions and the coset.

Let us consider this trace more closely. First, it is instructive to consider the trace without any GSO projector insertions. Taking into account the conditions on the NS-sector coset momenta, coming from the path integral construction of the coset partition function, *i.e.* the conditions $m - \bar{m} = 0$ and $m + \bar{m} = w \in \mathbb{Z}_2$, and obtaining the R-sector by 1/2-spectral flow, gives

$$\begin{aligned}
\frac{1}{4}\sum_{a,\bar{a}}\sum_{w\in\mathbb{Z}_2}(-1)^{a+\bar{a}}\Bigg\{\int_0^\infty & ds\sqrt{2}\rho(s,w;a,\bar{a};\epsilon)\,\times \\
& \chi_c\!\left(s,\frac{w+a}{2};\tau,0\right)\begin{bmatrix}a\\0\end{bmatrix}\chi_c\!\left(s,\frac{w+\bar{a}}{2};\bar{\tau},0\right)\begin{bmatrix}\bar{a}\\0\end{bmatrix} + \\
& + \frac{1}{2}\chi_d\!\left(\frac{w}{2},\frac{a}{2};\tau,0\right)\begin{bmatrix}a\\b\end{bmatrix}\chi_d\!\left(\frac{w}{2},\frac{\bar{a}}{2};\bar{\tau},0\right)\begin{bmatrix}\bar{a}\\0\end{bmatrix}\Bigg\}\,\times \\
\times\frac{1}{(8\pi^2\tau_2)^2\eta^2\bar{\eta}^2}\frac{\theta[^a_0]}{\eta}\frac{\theta[^{\bar{a}}_0]}{\bar{\eta}}\,. &
\end{aligned} \tag{B.12}$$

This sum contains a independent summation over the parameters $a, \bar{a}$ accounting for the NS/R-sectors, a summation over the $\mathrm{U}(1)_R$ charges of the $\mathcal{N} = 2$ primaries, and finally either an integration or a summation over the Casimir eigenvalue of the coset primaries. An extra minus sign in front of the R-NS or NS-R sectors accounts for spacetime statistics. This effect is responsible for the factor $(-1)^{a+\bar{a}}$.

Tracing over the Hilbert space with an insertion of $(-1)^{J_{\mathrm{GSO}}}$ yields similar results, but with characters having $b = 1$. In addition, an extra factor $(-1)^{\eta ab}$ selects the type-IIA or type-IIB GSO projection ($\eta = 1$ for type-IIA and $\eta = 0$ for type-IIB). The only subtlety is that since the definition of $b = 1$ characters for the coset involves the insertion of $(-1)^{F_c}$, where $Q_a = 2m_t + F_c$, a factor $(-1)^{2bm_t} = (-1)^{b(w+a)}$ remains explicit. Finally, an extra factor of $(-1)^{b(a+1)}$ accounts for the superghost contribution to J_{GSO}. Putting everything together and summing over $b, \bar{b} = 0, 1$ yields the type-II partition sum (2.25).

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